

Common Fixed- Point Theorems For (F, ψ, ϕ) -Contractions In Bipolar S_b -Metric Spaces And Application To Volterra Integral Equations

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Abstract

In this article, we introduce the concept of bipolar S_b -metric spaces as a generalization of bipolar metric spaces [9] and F -bipolar metric spaces [17]. Two common fixed-point theorems are established for four self-mappings satisfying (F, ψ, ϕ) -contraction. The first theorem is proved by using the common limit in the range (CLR) property [16, 22] without requiring continuity of the mappings or completeness of the space, while the second theorem is proved in the framework of complete bipolar S_b -metric spaces. Some corollaries are obtained as direct consequences of our main results. Non-trivial examples and an application to Volterra integral equations are also provided.

Keywords: bipolar S -metric space, (F, ψ, ϕ) -contraction, common fixed point, CLR property, Volterra integral equations.

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1 Introduction

The Banach contraction principle [1] is one of the most influential results in fixed point theory, ensuring existence and uniqueness of fixed points for contractive mappings in complete metric spaces. It has been extended in many directions. Notable generalizations include b -metric spaces due to Czerwik [2], Kannan-type contractions [3], altering distance functions introduced by Khan et al. [4], partial metric spaces due to Matthews [5], statistical metric spaces [6], fuzzy metric and statistical metric spaces [7], and fixed points in fuzzy metric spaces [8]. In 2016, Mutlu and Grdal [9] introduced bipolar metric spaces where the metric is defined on a pair of disjoint sets. This structure has been widely studied. Mutlu et al. [10, 12] proved coupled fixed -point theorems and local and weak contraction principles, Rao et al. [11] studied Geraghty-type contractions with applications to homotopy theory, and Kishore et al. [13] obtained common coupled fixed -point results and applications. In parallel, Shukla [15] introduced partial rectangular metric spaces, Rawat et al. [17] introduced F -bipolar metric spaces, and Paul and Gairola [18, 19, 21] studied rational type and

generalized weak contractions in F -metric spaces. The common limit in the range (CLR) property introduced by Faraji et al. [16] is a useful tool to prove common fixed- point theorems without completeness or continuity, and it was recently used in bipolar metric spaces by Rao et al. [22]. Stability of fixed- point problems including well-posedness and Ulam-Hyers stability was studied by Bota et al. [14], Demma et al. [20] and Saha et al. [23]. Applications to fuzzy systems were discussed in [24, 25].

In this article, we introduce bipolar S_b -metric spaces as a natural generalization of bipolar metric spaces [9] and F -bipolar metric spaces [17]. We introduce (F, ψ, ϕ) -contraction for four self-mappings and establish two common fixed point theorems. Theorem 1 is proved by using the joint $CLR_{S,T}$ property [16, 22] and weak compatibility [22] without requiring completeness of the space or continuity of the mappings. Theorem 2 is proved in the setting of complete bipolar S_b -metric spaces. Our results extend and improve several results in [9, 12, 17, 22]. Some corollaries are derived, two illustrative examples with graphs are given, and an application to a system of Volterra integral equations is presented.

2 Preliminaries

In this section, we recall some necessary definitions and state our proposed structure.

Definition 1 (Bipolar S_b -metric [11, 17]). Let X and Y be two non-empty sets. A mapping $S_b: (X \times X \times Y) \cup (X \times Y \times Y) \rightarrow [0, \infty)$ is called bipolar S_b -metric if for every $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$,

1. $S_b(x_1, x_2, y_1) \geq 0$, $S_b(x_1, y_1, y_2) \geq 0$,
2. $S_b(x_1, x_2, y_1) = 0$ if and only if $x_1 = x_2 = y_1$ and $S_b(x_1, y_1, y_2) = 0$ if and only if $x_1 = y_1 = y_2$,
3. $S_b(x, x, y) = S_b(y, y, x)$ for all $(x, y) \in X \times Y$,
4. $S_b(x_1, x_2, y_1) \leq S_b(x_1, x_1, y_2) + S_b(x_2, x_2, y_2) + S_b(x_1, x_2, y_1)$ and $S_b(x_1, y_1, y_2) \leq S_b(x_1, x_1, y_3) + S_b(x_3, y_1, y_2) + S_b(x_3, y_2, y_2)$.

Then (X, Y, S_b) is called a bipolar S_b -metric space.

Example 1. Let $X = [0, 1]$, $Y = [0, 1]$ and define $S_b: X \times Y \times Y \rightarrow [0, \infty)$ by $S_b(x, y_1, y_2) = |x - y_1| + |y_1 - y_2| + |x - y_2|$, for all $x \in X$, $y_1, y_2 \in Y$. Then (X, Y, S_b) is a bipolar S_b -metric space. Moreover, it is complete.

Definition 2 (Bisequence and biconvergence [9]). Let (X, Y, S_b) be a bipolar S_b -metric space.

- A sequence $\{x_n\}$ in X left-converges to $y \in Y$, written $x_n \rightarrow^l y$, if $\lim_{n \rightarrow \infty} S_b(x_n, x_n, y) = 0$.
- A sequence $\{y_n\}$ in Y right-converges to $x \in X$, written $y_n \rightarrow^r x$, if $\lim_{n \rightarrow \infty} S_b(x, x, y_n) = 0$.
- A pair $\langle \{x_n, y_n\} \rangle$ with $x_n \in X$, $y_n \in Y$ is a bisequence. It is biconvergent to $z \in X \cap Y$ if $x_n \rightarrow^l z$ and $y_n \rightarrow^r z$.

Definition 3 (Cauchy bi-sequence and completeness [9, 10]). A bisequence $\langle \{x_n, y_n\} \rangle$ in (X, Y, S_b) is Cauchy if for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $S_b(x_n, x_m, y_k) < \varepsilon$ for all $n, m, k \geq N$. (X, Y, S_b) is complete if every Cauchy bisequence is biconvergent in $X \cap Y$.

Definition 4 (Class $\mathcal{F}$ ([17])). Let $\mathcal{F}$ be the family of functions $F: (0, \infty) \rightarrow \mathbb{R}$ satisfying:

1. F is strictly increasing,
2. $\lim_{t \rightarrow 0^+} F(t) = -\infty$,
3. There exists $k \in (0, 1)$ such that $\lim_{t \rightarrow 0^+} t^k F(t) = 0$.

Definition 5 (Altering distance). A function $\psi: [0, \infty) \rightarrow [0, \infty)$ is an altering distance if it is continuous, non-decreasing and $\psi(t) = 0$ iff $t = 0$. A function $\phi: [0, \infty) \rightarrow [0, \infty)$ is said to be lower semi-continuous from right with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$.

Definition 6 ((F, ψ, ϕ) -contraction). Let (X, Y, S_b) be a bipolar S_b -metric space. Four mappings $A, B, S, T: X \cup Y \rightarrow X \cup Y$ are said to satisfy

(F, ψ, ϕ) -contraction if there exist $F \in \mathcal{F}$, $\tau > 0$ such that for all $x \in X$, $y \in Y$ with $S_b(Ax, By, By) > 0$, $\tau + F(\psi(S_b(Ax, By, By))) \leq F(\psi(M(x, y) - \phi(M(x, y))))$, where $M(x, y) =$

$$\max \left\{ \begin{array}{l} S_b(Sx, Ty, Ty), \\ S_b(Ax, Sx, Sx), \\ S_b(By, Ty, Ty), \\ \frac{S_b(Ax, Ty, Ty) + S_b(By, Sx, Sx)}{2} \end{array} \right\}$$

Definition 7 (Joint $CLR_{S,T}$ property ([16, 22])). Let A, B, S, T be as above. The pairs (A, S) and (B, T) satisfy joint common limit in the range of S and T property if one can find bisequences $\{x_n\} \subset X$, $\{y_n\} \subset Y$ such that $\lim_{n \rightarrow \infty} S_b(Ax_n, Ax_n, z) = \lim_{n \rightarrow \infty} S_b(Sx_n, Sx_n, z) = \lim_{n \rightarrow \infty} S_b(By_n, By_n, z) = \lim_{n \rightarrow \infty} S_b(Ty_n, Ty_n, z) = 0$ for some $z \in S(X) \cap T(Y)$.

Definition 8 (Weak compatibility ([22])). Two mappings $P, Q: X \cup Y \rightarrow X \cup Y$ are weakly compatible if $PQx = QPx$ whenever $Px = Qx$.

3 Main Results

Theorem 1. Let (X, Y, S_b) be a bipolar S -metric space. Let $A, B, S, T: X \cup Y \rightarrow X \cup Y$ be four self mappings satisfying the following contraction condition:

$$\tau + F(\psi(S_b(Ax, By, By))) \leq F(\psi(M(x, y) - \phi(M(x, y)))) \dots \dots \dots (i)$$

for all $x \in X$ and $y \in Y$ with $S_b(Ax, By, By) > 0$, where $\tau > 0$, $F \in \mathcal{F}$, ψ and ϕ are two altering function and $M(x, y) =$

$$\max \left\{ \begin{array}{l} S_b(Sx, Ty, Ty), \\ S_b(Ax, Sx, Sx), \\ S_b(By, Ty, Ty), \\ \frac{S_b(Ax, Ty, Ty) + S_b(By, Sx, Sx)}{2} \end{array} \right\} \dots \dots \dots (ii)$$

Suppose that

1. The pairs (A, S) and (B, T) satisfy the joint $CLR_{S,T}$ property, i.e. there exist sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = z$, for some $z \in S(X) \cap T(Y)$,
2. The pairs (A, S) and (B, T) are weakly compatible (i.e. they commute at their coincidence points).

Then A, B, S and T have a unique common fixed point in $X \cap Y$, i.e. there exists a unique point z such that $Az = Bz = Sz = Tz = z$.

Proof. Step 1: Existence of Coincidence Points.

From $CLR_{S,T}$ property we have sequences as in (1), Since $z \in S(X)$, $\exists u \in X$ s.t. $Su = z$. Also $z \in T(Y)$, $\exists v \in Y$ s.t. $Tv = z$.

Claim: $Au = z$. Assume $Au \neq z$ so $S_b(Au, z, z) > 0$. Put $x = u$ and $y = y_n$ in (i) and (ii)

$$\begin{aligned} & \tau + F(\psi(S_b(Au, By_n, By_n))) \\ & \leq F(\psi(M(u, y_n) - \phi(M(u, y_n)))). \end{aligned}$$

Where

$$= \max \left\{ \begin{array}{l} M(u, y_n) \\ S_b(Su, Ty_n, Ty_n), \\ S_b(Au, Su, Su), \\ S_b(By_n, Ty_n, Ty_n), \\ \frac{S_b(Au, Ty_n, Ty_n) + S_b(By_n, Su, Su)}{2} \end{array} \right\}$$

Taking $n \rightarrow \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} M(u, y_n) &= \max \left\{ \begin{array}{l} S_b(z, z, z), \\ S_b(Au, z, z), \\ S_b(z, z, z), \\ \frac{S_b(Au, z, z) + S_b(z, z, z)}{2} \end{array} \right\} \\ &= S_b(Au, z, z). \end{aligned}$$

Hence

$$\begin{aligned} & \tau + F(\psi(S_b(Au, z, z))) \\ & \leq F(\psi(S_b(Au, z, z) \\ & \quad - \phi(S_b(Au, z, z)))). \end{aligned}$$

Since $\phi(t) > 0$ and $\phi(t) < t$, we have $S_b(Au, z, z) - \phi(S_b(Au, z, z)) < S_b(Au, z, z)$. As F is strictly increasing, $\tau + F(\psi(S_b(Au, z, z))) < F(\psi(S_b(Au, z, z)))$ which implies $\tau < 0$, contradiction as $\tau > 0$.

Thus $S_b(Au, z, z) = 0 \Rightarrow Au = z$. Therefore $Au = Su = z$.

Similarly taking $x = x_n$ and $y = v$ we get $Bv = Tv = z$.

Step 2: Finding the Common Fixed Point via Weak Compatibility. Since (A, S) is weakly compatible and u is their coincidence point,

$$ASu = SAu \Rightarrow Az = Sz.$$

Suppose $Az \neq z$, so $S_b(Az, z, z) > 0$. Put $x = z$ and $y = v$ in the contraction (i) and (ii);

$$M(z, v) = \max \left\{ \begin{array}{l} S_b(Sz, Tv, Tv), \\ S_b(Az, Sz, Sz), \\ S_b(Bv, Tv, Tv), \\ \frac{S_b(Az, Tv, Tv) + S_b(Bv, Sz, Sz)}{2} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} S_b(Sz, z, z), \\ S_b(Az, Az, Az), \\ S_b(z, z, z), \\ \frac{S_b(Az, z, z) + S_b(z, Az, Az)}{2} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} S_b(Az, z, z), \\ 0, \\ 0, \\ S_b(Az, z, z) \end{array} \right\}$$

$$= S_b(Az, z, z)$$

$$[\text{as } S_b(Az, z, z) = S_b(z, Az, Az)]$$

Then

$$\begin{aligned} & \tau + F(\psi(S_b(Az, Bv, Bv))) \leq F\left(\psi\left(\frac{S_b(Az, z, z) - \phi(S_b(Az, z, z))}{2}\right)\right) \\ & \Rightarrow \tau + F(\psi(S_b(Az, z, z))) < \\ & F(\psi(S_b(Az, z, z))) \\ & \Rightarrow \tau < 0 \end{aligned}$$

Which is a Contradiction as $\tau > 0$.

So our assumption is Wrong,

$$\begin{aligned} & \text{So } S_b(Az, z, z) = 0 \\ & \Rightarrow Az = z. \end{aligned}$$

. Therefore,

$$Az = Sz = z.$$

By applying the exact same logic to the weakly compatible pair (B, T) at their meeting point v , and given contradiction,

$$Bz = Tz = z.$$

Combining both results, we get,

$$Az = Bz = Sz = Tz = z.$$

So, z is a Common fixed point of A, B, S and T .

Step 3: Uniqueness of the Common Fixed Point.

Let us assume that w be another fixed point s.t. $Aw = Sw = Tw = Bw = w$ and $w \neq z$, so $S_b(z, w, w) > 0$. By substituting $x = z$ and $y = w$ into the contractive condition, we get

$$\begin{aligned} & \tau + F(\psi(S_b(Az, Bw, Bw))) \\ & \leq F(\psi(M(z, w) - \phi(M(z, w)))). \\ & \Rightarrow \tau + F(\psi(S_b(z, w, w))) \\ & \leq F(\psi(M(z, w) - \phi(M(z, w)))). \end{aligned}$$

Now,

$$\begin{aligned} & M(z, w) \\ &= \max \left\{ \begin{array}{l} S_b(Sz, Tw, Tw), \\ S_b(Az, Sz, Sz), \\ S_b(Bw, Tw, Tw), \\ \frac{S_b(Az, Tw, Tw) + S_b(Bw, Sz, Sz)}{2} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} S_b(z, w, w), \\ S_b(z, z, z), \\ S_b(w, w, w), \\ \frac{S_b(z, w, w) + S_b(w, z, z)}{2} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} S_b(z, w, w), \\ 0, \\ 0, \\ S_b(z, w, w) \end{array} \right\} \\ &= S_b(z, w, w). \quad [\text{as}] \end{aligned}$$

$$S_b(z, w, w) = S_b(w, z, z)$$

So we have, the contractive condition,

$$\begin{aligned} & \tau + F(\psi(S_b(z, w, w))) \\ & \leq F(\psi(S_b(z, w, w) \\ & \quad - \phi(S_b(z, w, w)))). \end{aligned}$$

$$[\text{as } \phi(t) > 0 \text{ and } \phi(t)$$

$$< t \text{ So } S_b(z, w, w)$$

$$- \phi(S_b(z, w, w)) < S_b(z, w, w)]$$

$$\Rightarrow \tau + F(\psi(S_b(z, w, w))) < F(\psi(S_b(z, w, w)))$$

$$\Rightarrow \tau < 0$$

Which is a Contradiction as $\tau > 0$.

So our assumption is wrong.

So $S_b(z, w, w) = 0 \Rightarrow z = w$.

Hence, the four function A, B, S, T has a Common unique fixed Point. $\square$

Theorem 2 (Completeness version without CLR Property). Let (X, Y, S_b) be a Complete Bipolar S-metric space. Let $A, S, B, T: X \cup Y \rightarrow X \cup Y$ be four self-mappings satisfying the following Contractive Condition:

$$\tau + F(\psi(S_b(Ax, By, By))) \leq$$

$$F(\psi(M(x, y) - \phi(M(x, y)))) \dots\dots\dots (iii)$$

for all $x \in X$ and $y \in Y$ whenever $S_b(Ax, By, By) > 0$, where $\tau > 0$, $F \in \mathcal{F}$, ψ and ϕ are two altering function and $M(x, y) =$

$$\max \left\{ \begin{array}{l} S_b(Sx, Ty, Ty), \\ S_b(Ax, Sx, Sx), \\ S_b(By, Ty, Ty), \\ \frac{S_b(Ax, Ty, Ty) + S_b(By, Sx, Sx)}{2} \end{array} \right\} \dots\dots\dots (iv)$$

Suppose that:

1. $A(X) \subseteq T(Y)$ and $B(Y) \subseteq S(X)$,
2. One of the Subspaces $A(X), B(Y), S(X)$, or $T(Y)$ is closed subset of X or Y .
3. The Pairs (A, S) and (B, T) are Weakly Compatible.

Then A, B, S, T have a unique Common fixed point in $X \cap Y$ i.e. there exists a unique point $z \in X \cap Y$ such that $Az = Bz = Sz = Tz = z$

Proof. Step 1:- (Construction of the Iterative sequences)

Let $x_0 \in X$ be arbitrary Point. Since $A(X) \subseteq T(Y)$, there exists a point $y_0 \in Y$ such that $Ax_0 = Ty_0$.

Since $B(Y) \subseteq S(X)$, there exists a Point $x_1 \in X$ such that $By_0 = Sx_1$.

By Induction, We can Construct two sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ such that

$Sx_{2n+1} = By_{2n}$ and $Ty_{2n+1} = Ax_{2n+2}$ for $n \geq 0$.

For Convenience, let

$$d_{2n} = S_b(Ax_{2n}, By_{2n-1}, By_{2n-1}) \text{ and } d_{2n+1} = S_b(Ax_{2n}, By_{2n+1}, By_{2n+1})$$

and we assume $d_n > 0 \forall n \in \mathbb{N}$. (If $d_n = 0$ for any n , a Common fixed Point is immediately found)

Step 2: (Proving the sequences are Cauchy)

By Substituting $x = x_{2n}$ and $y = y_{2n-1}$, into the main Contractive Condition, we get;

$$\begin{aligned} \tau + F(\psi(S_b(Ax_{2n}, By_{2n-1}, By_{2n-1}))) \\ \leq F(\psi(M(x_{2n}, y_{2n-1}) - \phi(M(x_{2n}, y_{2n-1})))) \\ \Rightarrow \tau + F(\psi(d_{2n})) \\ \leq F(\psi(M(x_{2n}, y_{2n-1}) - \phi(M(x_{2n}, y_{2n-1})))) \end{aligned}$$

Now,

$$\begin{aligned} & M(x_{2n}, y_{2n-1}) \\ &= \max \left\{ \begin{array}{l} S_b(Sx_{2n}, Ty_{2n-1}, Ty_{2n-1}), \\ S_b(Ax_{2n}, Sx_{2n}, Sx_{2n}), \\ S_b(By_{2n-1}, Ty_{2n-1}, Ty_{2n-1}), \\ \frac{S_b(Ax_{2n}, Ty_{2n-1}, Ty_{2n-1}) + S_b(By_{2n-1}, Sx_{2n}, Sx_{2n})}{2} \end{array} \right\} \\ &= \\ & \max \left\{ \begin{array}{l} S_b(By_{2n-1}, Ax_{2n}, Ax_{2n}), \\ S_b(Ax_{2n}, By_{2n-1}, By_{2n-1}), \\ S_b(By_{2n-1}, Ax_{2n}, Ax_{2n}), \\ \frac{S_b(Ax_{2n}, Ax_{2n}, Ax_{2n}) + S_b(By_{2n-1}, By_{2n-1}, By_{2n-1})}{2} \end{array} \right\} \\ &= \\ & \max\{d_{2n-1}, d_{2n}, d_{2n-1}, 0\} \\ &= \max\{d_{2n}, d_{2n-1}\} \end{aligned}$$

If $\max\{d_{2n}, d_{2n-1}\} = d_{2n}$,

Then the equation (iii) becomes,

$$\begin{aligned} \tau + F(\psi(d_{2n})) &\leq F(\psi((d_{2n}) - \phi(d_{2n}))) \\ \text{since } \phi(t) > 0 \quad \forall t > 0 \text{ and } \phi(t) < t \\ &\Rightarrow \psi(d_{2n}) - \phi(d_{2n}) < \psi(d_{2n}) \\ &\Rightarrow \tau + F(\psi(d_{2n})) < F(\psi(d_{2n})) \\ &\Rightarrow \tau < 0 \end{aligned}$$

Which is a Contradiction.

So $M(x_{2n}, y_{2n-1}) = \max\{d_{2n}, d_{2n-1}\} = d_{2n-1}$

Substituting in equation (iii) gives

$$\begin{aligned} \tau + F(\psi(d_{2n})) &\leq F(\psi((d_{2n-1}) - \phi(d_{2n}))) \\ &\Rightarrow \tau + F(\psi(d_{2n})) < F(\psi(d_{2n-1})) \end{aligned}$$

Since F and ψ are strictly monotone increasing, then (iv) implies that $d_{2n} < d_{2n-1} \forall n \in \mathbb{N}$.

By similar argument with $x = x_{2n}$ and $y = y_{2n+1}$, we have

$$d_{2n+1} < d_{2n} \quad \forall n \in \mathbb{N}.$$

Therefore $\{d_n\}$ is a strictly decreasing sequence of positive real number and bounded below.

So $\{d_n\}$ is Convergent.

Let $\lim_{n \rightarrow \infty} d_n = r$.

If $r > 0$, then by (4) we get and Taking $n \rightarrow \infty$ we get

$$\begin{aligned} \tau + F(\psi(r)) &< F(\psi(r)) \\ &\Rightarrow \tau < 0 \end{aligned}$$

Which is a Contradiction.

So $r = 0$.

$$\therefore \lim_{n \rightarrow \infty} d_n = 0.$$

Next we prove that the bisequence $\langle (Ax_n, By_n) \rangle$ be Cauchy.

By contractive Condition (iv) to generalize this expression for any arbitrary index, we substitute $2n$ with n and $2n - 1$ with $n - 1$. Then, we have

$$F(\psi(d_n)) < F(\psi(d_{n-1})) - \tau$$

For $n = 1$

$$F(\psi(d_1)) < F(\psi(d_0)) - \tau$$

For $n = 2$

$$\begin{aligned} F(\psi(d_2)) &< F(\psi(d_1)) - \tau \\ &< F(\psi(d_0)) - 2\tau \end{aligned}$$

For n times

$$F(\psi(d_n)) < F(\psi(d_0)) - n\tau.$$

Multiply $[\psi(d_n)]^k$ where $0 < k < 1$ from both sides we get,

$$\begin{aligned} & [\psi(d_n)]^k F(\psi(d_n)) \\ & < [\psi(d_n)]^k F(\psi(d_0)) \\ & - [\psi(d_n)]^k n\tau \\ \Rightarrow & n\tau [\psi(d_n)]^k < [\psi(d_n)]^k F(\psi(d_0)) \\ & - [\psi(d_n)]^k F(\psi(d_n)) \end{aligned}$$

Taking $n \rightarrow \infty$ Then $[\psi(d_n)]^k F(\psi(d_n)) \rightarrow 0$ and $F(\psi(d_0))$ is constant and $[\psi(d_n)]^k \rightarrow 0$ as $\psi(d_n) \rightarrow \psi(0) = 0$.

So $n\tau [\psi(d_n)]^k \rightarrow 0$, i.e.

$$\lim_{n \rightarrow \infty} n[\psi(d_n)]^k = 0 \quad \text{as } \tau > 0.$$

Then $\exists n_1 \in \mathbb{N}$ s.t

$$\begin{aligned} & n[\psi(d_n)]^k \leq 1 \quad \forall n > n_1 \\ \Rightarrow & \psi(d_n) \leq \frac{1}{n^{1/k}} \quad \forall n > n_1 \end{aligned}$$

Since ψ is Continuous and we Consider $\psi(t) > t$

Then we get $d_n < \psi(d_n) \leq \frac{1}{n^{1/k}} \quad \forall n > n_1$

$$\begin{aligned} & \Rightarrow d_n < \frac{1}{n^{1/k}} \quad \forall n > n_1 \\ \Rightarrow & \sum_{n=1}^{\infty} d_n < \sum_{n=1}^{\infty} \frac{1}{n^{1/k}} \text{ is Convergent} \end{aligned}$$

by p-test. So by Comparison Test $\sum_{n=1}^{\infty} d_n$ is Convergent.

So for $m > n > n_1$, by Triangle inequality,

$$\begin{aligned} S_b(Ax_n, By_m, By_m) & \leq S_b(Ax_n, By_{m-1}, By_{m-1}) \\ & + S_b(Ax_{n-1}, By_{m-1}, By_{m-1}) \\ & + S_b(Ax_{n-1}, By_m, By_m) \\ & \leq \sum_{i=n}^{m-1} S_b(Ax_i, By_{i-1}, By_{i-1}) \\ & = \sum_{i=n}^{m-1} d_i \end{aligned}$$

Since $\sum_{n=1}^{\infty} 1/n^{1/k}$ converges for $1/k > 1$, its tail $\sum_{i=n}^{m-1} 1/i^{1/k} \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\lim_{m,n \rightarrow \infty} S_b(Ax_n, By_m, By_m) = 0.$$

Hence the bisequence $\langle (Ax_n, By_n) \rangle$ be Cauchy and (X, Y, S_b) be complete.

So both sequences converges to the same intersection point $z \in X \cap Y$.

$$\text{So } \lim Ax_n = \lim By_n = \lim Sx_n = \lim Ty_n = z$$

Step-3 :- Existence of Coincidence points

Since $z \in X$ and $S(X)$ is closed and $\lim Sx_n = z$, Then $z \in S(X)$.

Then there must be exist $u \in X$ s.t

$$Su = z.$$

Also $z \in Y$ and $T(Y)$ is closed and $\lim Ty_n = z$, then $z \in T(Y)$

Then there must be exist $v \in Y$ s.t

$$Tv = z.$$

Proof of the Remaining part of Theorem 2

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1.

Since Theorem 3.1 has already been proved in the paper, and the construction of the Cauchy bisequence $\langle (Ax_n, By_n) \rangle$, its convergence to $z \in X \cap Y$, and the existence of $u \in X, v \in Y$ such that $Su = z$ and $Tv = z$ have been established above, the proof of existence of coincidence and common fixed point follows on the same lines as in Steps 3-5 of Theorem 1. Hence proved. $\square$

Corollary 3. Let $\psi(t) = t$ in Theorem 1, the condition becomes $\tau + F(S_b(Ax, By, By)) \leq F(M(x, y) - \phi(M(x, y)))$ Where $M(x, y) =$

$$\sup \left\{ \begin{array}{l} S_b(Sx, Ty, Ty), \\ S_b(Ax, Sx, Sx), \\ S_b(By, Ty, Ty), \\ \frac{S_b(Ax, Ty, Ty) + S_b(By, Sx, Sx)}{2} \end{array} \right\} \text{ Then } A, B, S, T \text{ have a}$$

unique common fixed point.

Remark 1. This shows ψ is not essential, it generalizes the contraction.

Corollary 4. Let $\phi(t) = 0$ in Theorem 1, the condition becomes

$$\tau + F(\psi(S_b(Ax, By, By))) \leq$$

$$F(\psi(M(x, y)))$$

Then A, B, S, T have a unique common fixed point.

Remark 2. This is the F_ψ -contraction in bipolar S_b -metric space.

Corollary 5. Let $F(t) = \ln t, \psi(t) = t, \phi(t) = (1 - k)t$ with $0 < k < 1$. Then Theorem 1 reduces to

$$\begin{aligned} S_b(Ax, By, By) & \leq ke^{-\tau} M(x, y) \\ & \leq k_1 M(x, y) \end{aligned}$$

Where $k_1 = ke^{-\tau} \in (0, 1)$. Hence A, B, S, T have a unique common fixed point.

Remark 3. This is the Banach Type Contraction.

Corollary 6. Let $A = B = P$ and $S = T = Q$ in Theorem 3.1. Then two mappings $P, Q: X \cup Y \rightarrow X \cup Y$

$$\tau + F(\psi(S_b(Px, Qy, Qy))) \leq F(\psi(M_{P,Q}(x, y) - \phi(M_{P,Q}(x, y))))$$

$$\text{Where, } M_{P,Q}(x, y) = \max \left\{ \begin{array}{l} S_b(Sx, Ty, Ty), \\ S_b(Px, Sx, Sx), \\ S_b(Qy, Ty, Ty), \\ \frac{S_b(Px, Ty, Ty) + S_b(Qx, Sx, Sx)}{2} \end{array} \right\}.$$

Then P and Q have a unique common fixed point.

4 Examples

Example 2. Let $X = [0, 1], Y = [0, 1]$ and we consider bipolar S_b -metric space $S_b(x, y, z) = |x - y| + |y - z| + |z - x|$. This is a complete bipolar S_b -metric space.

Define mappings: $Sx =$
 $\begin{cases} x, & 0 \leq x \leq 0.5 \\ 0.5, & 0.5 \leq x \leq 1 \end{cases}$
 $Tx = \begin{cases} x, & 0 \leq x \leq 0.5 \\ 0.5, & 0.5 \leq x \leq 1 \end{cases}$
 $Ax = \begin{cases} 0, & 0 \leq x \leq 0.5 \\ 0.25, & 0.5 < x \leq 1 \end{cases}$
 $Bx = \begin{cases} 0, & 0 \leq x \leq 0.5 \\ 0.25, & 0.5 < x \leq 1 \end{cases}$
Take $F(t) = \ln t + t$, $\psi(t) = 2t$, $\phi(t) = \frac{t}{3}$, $\tau = 0.5$.

Verification:

(i) $S(X) = [0, 0.5]$, $T(Y) = [0, 0.5]$ closed. $A(X) = \{0, 0.25\} \subset T(Y)$ $B(Y) = \{0, 0.25\} \subset S(X)$.

(ii) Common fixed point: $S0 = 0$, $A0 = 0$, $T0 = 0$, $B0 = 0$. So 0 is a Common fixed Point.

(iii) Contraction check

For $x, y \in [0, 0.5]$, $S_b(Ax, By, By) = 0$. So Condition holds.

For $x, y \in (0.5, 1]$ $S_b(Ax, By, By) =$

$S_b(0.25, 0.25, 0.25) = 0$ So condition holds.

For $x \in [0, 0.5]$, $y \in (0.5, 1]$

$$\begin{aligned} S_b(Ax, By, By) &= S_b(0, 0.25, 0.25) \\ &= |0 - 0.25| + |0.25 - 0.25| + |0.25 - 0| \\ &= 0.25 + 0.25 \\ &= 0.5 \end{aligned}$$

$$S_b(Sx, Ty, Ty) = S_b(x, 0.5, 0.5)$$

$$= 2|x - 0.5|$$

$$S_b(Ax, Sx, Sx) = S_b(0, x, x)$$

$$= 2x$$

$$S_b(By, Ty, Ty) = S_b(0.25, 0.5, 0.5)$$

$$= 0.5$$

$$\frac{S_b(Ax, Ty, Ty) + S_b(By, Sx, Sx)}{2} = \frac{1 + 0.2}{2} = \frac{1.2}{2} = 0.6$$

So $M(x, y) = \max\{0.5, 2|x - 0.5|, 2x, 0.6\} = 1$

Hence $S_b(Ax, By, By) = 0.5 \leq M(x, y)$

$$L.H.S = \tau + F(\psi(0.5))$$

$$= 0.5 + \ln(1) + 1 = 1.5$$

$$R.H.S = F(\psi(M - \phi(M))) = F\left(2\left(M - \frac{M}{3}\right)\right) = F\left(\frac{4M}{3}\right)$$

$$= \ln\left(\frac{4}{3}\right) + \frac{4}{3}$$

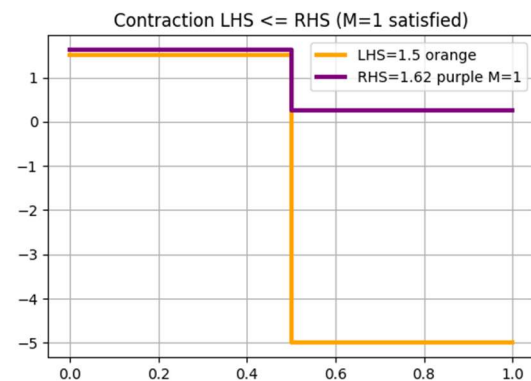
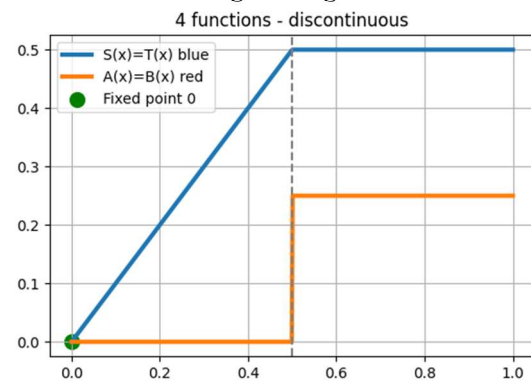
$$= 1.62$$

So $\tau + F(\psi(S_b(Ax, By, By))) \leq F(\psi(M(x, y) - \phi(M(x, y))))$

Hence Condition holds.

(iv) Coincidence points are at $x \in [0, 0.5]$ where $Ax = Sx = x$ only at $x = 0$. $AS(0) = 0 = SA(0)$. Similarly (B, T) also satisfy Weakly Compatible Condition.

Therefore 0 is the unique Common fixed Point for functions A, B, S, T .



Verification of Example 2, common fixed point 0. S, T, A, B are discontinuous at 0.5 and $\tau + F(\psi(S_b)) \leq F(\psi(M - \phi(M)))$ for $M = 1$

Example 3 (Non-complete bipolar S_b -metric). Let $X = (0, 1]$, $Y = [0, 1]$ and $S_b(x, y, z) = |x - y| + |y - z| + |z - x|$ be a bipolar S_b -metric. Then (X, Y, S_b) is **not complete**, since $\langle x_n \rangle = \langle \frac{1}{n} \rangle$ in X and $\langle y_n \rangle = \langle \frac{1}{n} \rangle$ in Y is Cauchy but $\langle \frac{1}{n} \rangle \rightarrow 0 \notin X$ (i.e., not convergent).

Define four mappings $A, B, S, T: X \cup Y \rightarrow X \cup Y$ by $Ax = 1$, $Bx = 1$, $Sx = 1$, $Tx = 1$, $\forall x \in X, y \in Y$. Define $F(t) = \ln t$, $\psi(t) = t$, $\phi(t) = t/2$, and $\tau = \ln 2$.

Verification. (i) **CLR Property:** Take sequence $\langle x_n \rangle = \langle \frac{1}{n} \rangle$ in X

$$\lim_{n \rightarrow \infty} Sx_n = 1 \in X \cap Y, \quad \lim_{n \rightarrow \infty} Ax_n = 1 \in X \cap Y.$$

So $\lim Sx_n = \lim Ax_n = 1 \in X \cap Y$. Hence (S, A) satisfies $CLR_{S,A}$ property.

Similarly take $\langle y_n \rangle = \langle \frac{1}{n} \rangle$ in Y ,

$$\lim_{n \rightarrow \infty} Ty_n = 1, \quad \lim_{n \rightarrow \infty} By_n = 1.$$

So (B, T) satisfies $CLR_{B,T}$ property.

(ii) **Contraction check:**

$$S_b(Ax, By, By) = S_b(1, 1, 1) = 0.$$

Then $\tau + F(\psi(0)) = \tau + F(0) = -\infty$, while

$$F(\psi(M - \phi(M))) = \ln\left(\frac{M}{2}\right) \in \mathbb{R}.$$

So $\tau + F(\psi(S_b(Ax, By, By))) \leq F(\psi(M(x, y) - \phi(M(x, y))))$ holds trivially.

(iii) **Common fixed point:** Clearly $1 \in X \cap Y$ and $A1 = B1 = S1 = T1 = 1$. So 1 is the unique common fixed point of A, B, S, T . $\square$

5 Application to Volterra integral equations

Let $X = C[0, T]$, $Y = C[0, T]$. Define bipolar S_b -metric by

$$S_b(x, y, z) = \sup_{t \in [0, T]} |x(t) - y(t)| + |y(t) - z(t)| + |z(t) - x(t)|$$

for $x \in X$, $y, z \in Y$, and $T \geq 1$. Then (X, Y, S_b) is a complete bipolar S_b -metric space.

We consider the system:

$$x(t) = g_1(t) + \int_0^t k_1(t, s, x(s)) ds \dots\dots\dots (v)$$

$$x(t) = g_2(t) + \int_0^t k_2(t, s, x(s)) ds \dots\dots\dots (vi)$$

for $t \in [0, T]$, $T \geq 1$.

Define mappings $S, T, A, B: X \cup Y \rightarrow X \cup Y$ by

$$Sx(t) = x(t)$$

$$Tx(t) = x(t)$$

$$Ax(t) = g_1(t) + \int_0^t k_1(t, s, x(s)) ds$$

$$Bx(t) = g_2(t) + \int_0^t k_2(t, s, x(s)) ds.$$

Assuming that:

1. $g_1(t) = g_2(t) \quad \forall t \in [0, T].$
2. $k_1, k_2: [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous.
3. $\exists F \in \mathcal{F}$, two altering functions ψ and ϕ be continuous and $\tau > 0$ such that $\forall x \in X$ and $y \in Y$ and $t, s \in [0, T], u, v \in \mathbb{R}$,

$$|k_1(t, s, u) - k_2(t, s, v)| \\ \leq \frac{1}{2T} \{ \psi^{-1} [F^{-1}(F(\psi(M(x, y) \\ - \phi(M(x, y)))) - \tau)] \}$$
 where $M(x, y)$ is as in Theorem 2.
4. $A(X) \subseteq T(Y)$, $B(Y) \subseteq S(X)$ and $S(X)$, $T(Y)$ are closed subsets of $X \cap Y$.

Then the system (v) and (vi) has a unique common solution $z \in X \cap Y$.

Proof. For any $x \in X$, $y \in Y$,

$$\begin{aligned} & \blacksquare(S_b(Ax, By, By)) \&= 2|Ax(t) - By(t)| \& \\ &= 2|\int_{0^+}^t [k_1(t, s, x(s)) \\ &\quad - k_2(t, s, y(s))] ds| \& \\ &\leq 2\int_{0^+}^t |k_1 - k_2| ds \quad ["by (iii)"] \& \\ &\leq 2\int_{0^+}^t 1/2T \{ \psi^{\wedge}(-1) [F^{\wedge}(-1) (F(\psi(M(x, y) \\ &\quad - \phi(M(x, y)))) - \tau)] \} ds \& \\ &\leq \psi^{\wedge}(-1) [F^{\wedge}(-1) (F(\psi(M(x, y) - \phi(M(x, y)))) \\ &\quad - \tau)] \end{aligned}$$

as $\frac{t}{T} \leq 1$ for $T \geq 1$ and $t \in [0, T]$.

Applying ψ both sides,

$$\begin{aligned} & \psi(S_b(Ax, By, By)) \\ & \leq F^{-1}(F(\psi(M(x, y) \\ & \quad - \phi(M(x, y)))) - \tau) \end{aligned}$$

Applying F both sides,

$$\begin{aligned} & \tau + F(\psi(S_b(Ax, By, By))) \\ & \leq F(\psi(M(x, y) - \phi(M(x, y)))) \end{aligned}$$

Then by Theorem 3.2, A, S, T, B has a unique common fixed point $z \in X \cap Y$ s.t. $Az = Bz = Sz = Tz = z$.

i.e., the system of Volterra integral Equations (v) and (vi) have a unique solution. $\square$

6 Conclusion

In this work, a new structure named bipolar S_b -metric space has been introduced, which extends the notions of bipolar metric [9] and F -bipolar metric [17]. Within this framework, two common fixed-point results for four self-mappings satisfying (F, ψ, φ) -contraction have been established.

The first result, Theorem 1, has been proved by employing the joint $CLR_{S,T}$ property [16,22] and weak compatibility ([22]). Its significance lies in the fact that completeness of the space and continuity of the mappings are not required. The second result, Theorem 2, has been established in the setting of complete bipolar S_b -metric spaces under the inclusions $A(X) \subseteq T(Y)$, $B(Y) \subseteq S(X)$ and closedness of ranges. Several corollaries have been derived as direct consequences of these theorems, showing that Banach and F -contraction type results are special cases.

To support the theoretical findings, two non-trivial examples equipped with graphical illustrations have been provided. Furthermore, an application to a system of Volterra integral equations has been given, which establishes the existence of a common solution and demonstrates the applicability of Theorem 2. The present results generalize and improve the corresponding results in [9,12,17,22].

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Data Availability

datasets were not generated or analyzed during the current study. All data are included in the manuscript.

Conflict of Interest

Commitment of interest:
The authors declare that they have no competing financial or non-financial interests.

Author Contributions

All authors contributed equally to conceptualization, formal analysis, writing of the original draft, and review and editing of the final version.

Ethical Approval

This article does not contain any studies with human or animal subjects performed by any of the authors.

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