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Full Length Article

Title: Existence And Uniqueness Of Solutions Of Integral Equations Via Fixed Point Theorems In Discrete G-Metric Spaces

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Abstract

The purpose of this paper is to introduce and study the Banach Fixed Point Theorem in the framework of ***G-metric spaces****, with particular emphasis on discrete G-metric spaces. The study establishes suitable conditions under which the existence and uniqueness of fixed points can be guaranteed. These fixed-point results are then applied to investigate the* ***existence and uniqueness of solutions for Fredholm and Volterra integral equations*** *in discrete G-metric spaces. The approach provides a useful extension of classical fixed-point techniques to a more general metric framework. Several illustrative examples are presented to demonstrate the applicability of the obtained results and to clarify the theoretical developments. The results may contribute to the further study of integral equations and fixed-point problems in generalized metric spaces.*

Keywords: *G-Metric Space, Integral Equation, Fixed Point, Volterra Equation, Fredholm Equation, Discrete G-Metric Space.*

AMS Subject Classification: 45D05, 47H10, 54H25.

Introduction:

Integral equation is central to the mathematical modelling of real-world phenomena, especially in physics, engineering biology and economics. In nonlinear analysis, the study of these equation's often hinges on the existence and uniqueness of solution.

In 2006, Zead Mustafa and Brailey Sims introduced a new structure of G-metric space (see [1]), and developed a new fixed-point theory for various mapping in this new structure.

In classical G-metric space, the condition for existence and uniqueness of solution of Volterra and Fredholm equation's is $|\lambda| < \frac{1}{3M(b-a)}$,but the aims of this paper is to introduce discreate G-metric space and update the condition $|\lambda| < \frac{1}{M(b-a)}$.

Preliminaries:

Definition 2.1 [1]: Let X be a non-empty set. A function $G: X \times X \times X \to R^{+}$ is called G-metric if:

- $G(x, y, z) = 0 \ iff \ x = y = z$
- $G(x, x, z) > 0 \ if \ x \neq y$
- $G(x, x, y) \leq G(x, y, z)$
- Symmetry in all 3 variables
- $G(x, y, z) \leq G(x, a, a) + G(a, y, z) \ \forall x, y, z \in X$

The pair (X, G) is called G-metric space.

Definition 2.2 ([1]). A G-metric space (X, G) is symmetric if $G(x, y, y) = G(x, x, y) \ \ \forall x, y \in X$.

Example 2.1. Let (X, d) be a metric space, and we define a G-metric

- $G_1(x, y, z) = Max\{d(x, y), d(x, z), d(y, z)\}$
- $G_2 = d(x, y) + d(x, z) + d(y, z)$

then $(X, G_1) and \ (X, G_2)$ are symmetric G-metric spaces.

The following definitions are about convergence and completeness on G-metric spaces.

Definition 2.3 ([1]). Let (X, G) be a G-metric space.

- A sequence $< x_n >$ in X, G-converges to x if and only if $G(x_n, x, x) \to 0 \ \ as \ n \to \infty$. That is for each $\epsilon > 0$there exists $n_0 \in N$ such that for all $n > n_0$, $G(x, x_n, x_n) < \in \ or \ G(x_n, x, x) < \in$.
- Sequence $< x_n >$ in X is called a G-Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $G(x_n, x_m, x_p) < \in$ for each $n, m, p > n_0$.
- The G-metric space (X, G) is said to be G-complete if every Cauchy sequence is G-convergent.

Proposition 2.1 ([1]): In a G-metric space (X, G)

- the sequence$< x_n >$ is G-Cauchy if and only if $\lim_{m,n\to\infty} G(x_n, x_n, x_m) = 0$;
- $G(x, x, y) \leq 2G(x, y, y)$;
- $G(x, y, z) = 0 \ iff \ x = y = z$

Proposition 2.2 ([1]). Let (X, G) be a G-metric space, then the function G (x, y, z) is jointly continuous in all three of its variables.

Definition 2.4 ([1]). Let (X, G) be a G-metric space and $T: X \to X$ be a self-map. T is said to be G-continuous if for any G-convergent sequence $< x_n >$ to x, then $< Tx_n >$ is G-convergent to T(x).

Main results

Banach's fixed Point Theorem

In 1992 Banach [2] proved a theorem, which is well known as "Banach's fixed point theorem "to establish the existence and uniqueness of solutions in integral equations.

Theorem 3.1:

Let (X, G) be a complete G-metric space. Let $T: X \to X$ be a mapping such that

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$G(Tx, Ty, Tz) \leq KG(x, y, z)\ \forall x, y \in X\ and\ 0 \leq K < 1.$

Then T has a unique fixed point.

Proof: choose an arbitrary point $x_0 \in X$. Define a sequence $< x_n >$ by

$$x_n = T(x_{n-1}) = T^2(x_{n-2}) = T^3(x_{n-3}) = T^4(x_{n-4}) = \cdots \ldots \ldots \ldots T^n(x_0) \quad \forall n \in N$$

Now apply the G-contractive condition to consecutive terms:

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \leq KG(x_{n-1}, x_n, x_n) \leq K^2 G(x_{n-2}, x_{n-1}, x_{n-1}) \leq \cdots \ldots \ldots \ldots \ldots$$
$$\leq K^n G(x_0, x_1, x_1)$$

Now we apply the G-metric properties for $m > n$

$$G(x_n, x_m, x_m) \leq \sum_{i=n}^{m-1} G(x_i, x_{i+1}, x_{i+1}) \leq (\sum_{i=n}^{m-1} K^i) \mathrm{G}(x_0, x_1, x_1) \leq \frac{K^n}{1-K} G(x_0, x_1, x_1)$$

Taking limit as $m, n \to \infty$ $then$ $G(x_n, x_m, x_m) \to 0$

Then we say that $< x_n >$ is a G-Cauchy sequence.

Existence of the limit point: Since (X, G) is complete, the G-Cauchy sequence $< x_n >$ converges to a limit $x^* \in X$.

By the continuity of the G-metric or T taking the limit of $x_{n+1} = Tx_n$ yield $Tx^* = x^*$,proving $x^*$is a fixed point.

Uniqueness: Let us assume $y^*$is another fixed point such that $Ty^* = y^*$. Consider $G(x^*, y^*, y^*) = G(Tx^*, Ty^*, Ty^*) \leq KG(x^*, y^*, y^*)$

Since $0 \leq k < 1$, then we have $\Longrightarrow (1-k)G(x^*, y^*, y^*) \leq 0 \ \Longrightarrow G(x^*, y^*, y^*) = 0 \Longrightarrow x^* = y^*$.

Application To Fredholm Integral Equation

Consider the Fredholm integral equation:

$$x(t) = f(t) + \lambda \int_a^b K(t, s) x(s) ds$$

Where x(t), K (s, t), f(t) all are belongs to $C[a, b]$.

Let us consider $X = C[a, b]$. the set of all continuous function, and define G-metric on X, where $G(x, y, z) = \begin{cases} 0 & if\ x(t) = y(t) = z(t) \\ 1, & otherwise \end{cases}$

, $t\epsilon[a, b]$.This G-Metric space is also known as Discreate G-metric space and also denoted by $G_{0,1}$.

Lemma 4.1: If $X = C[a, b]$ and $G(x, y, z) = G_{0,1}(x, y, z)$ then $(X, G_{0,1})$ is complete.

Proof: Let us consider $< x_n >$ be any Cauchy sequence in given $(X, G_{0,1})$ space.

Let for $\in > 0\ \exists s \in N\ and\ n, m, p \geq s\ \ then\ G_{0,1}(x_n, x_m, x_p) = 0$ and this imply $x_n = x_m = x_p$,so that the sequence $< x_n >$ is a constant sequence and convergent in constant value in X. So, Every Cauchy sequence in X also convergent in X and hence $(X, G_{0,1})$ is complete.

Theorem 4.1: Let $(X, G_{0,1})$ be a complete metric space where $X = C[a, b]$ and consider the Fredholm integral equation:

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$$x(t) = f(t) + \lambda \int_a^b K(t,s)x(s)ds \ldots\ldots\ldots\ldots\ldots.(i)$$

Where x(t), K (s, t), f(t) all are belongs to $C[a,b]$ and λ be constant.

If $(ii)|K(t,s)| \le M$ and $(iii)|\lambda| < \frac{1}{M(b-a)}$ then Fredholm integral equation has unique solution in $C[a,b]$.

Proof: At first, we define an operator $T: C[a,b] \to C[a,b]$ where $X = C[a,b]$ and let $Tx(t) = f(t) + \lambda \int_a^b K(t,s)x(s)ds \quad t \in [a,b]$.

Case-I: Let Tx=Ty=Tz.

Then if $Tx = Ty \Rightarrow f(t) + \lambda \int_a^b K(t,s)x(s)ds = f(t) + \lambda \int_a^b K(t,s)x(s)ds \Rightarrow M \lambda \int_a^b [x(s) - y(s)]ds = 0 \Rightarrow x(s) = y(s) where\ M \ne 0\ and\ \lambda \ne 0$.

Again if $Ty = Tz$ similarly to show that $y(s) = z(s)$.Therefor we have $x(s) = y(s) = z(s) \ldots\ldots\ldots\ldots..(iv)$

Now $G_{0,1}(Tx,Ty,Tz) = 0 = M|\lambda|[\int_a^b |(x(s) - y(s))|ds + \int_a^b |(y(s) - z(s))|ds + \int_a^b |(z(s) - x(s))|ds] = M|\lambda|(b-a).0 = M|\lambda|(b-a)G_{0,1}(x,y,z)$ using (ii) a (iv).

Then by the theorem 3.1 we say that T has a fixed point where $M|\lambda|(b-a). < 1\ i.e\ |\lambda| < \frac{1}{M(b-a)}$,and hence the Given integral equation (i) has a unique solution.

Case -II: Three contractive mapping are not equal i.e. Tx, Ty, Tz are not equal.

Then $|Tx(t) - Ty(t)| = |f(t) + \lambda \int_a^b K(t,s)x(s)ds - f(t) - \lambda \int_a^b K(t,s)y(s)ds| \le M|\lambda| | \int_a^b |x(s) - y(s)|ds \le M|\lambda|(b-a)Sup|x(s) - y(s)|$, using (ii)

As if $x \ne y$ then $Sup|x(s) - y(s)| > 0$ and if $M|\lambda|(b-a) < 1\ i.e\ |\lambda| < \frac{1}{M(b-a)}$ Then $Sup(|Tx - Ty)| < Sup(|x(s) - y(s)|$.

Therefor $G_{0,1}(Tx,Ty,Tz) \le M|\lambda|(b-a)G_{0,1}(x,y,z)$.

Then by the theorem 3.1 we say that T has a fixed point where $M|\lambda|(b-a). < 1\ i.e\ |\lambda| < \frac{1}{M(b-a)}$,and hence the Given integral equation (i) has a unique solution.

Example 4.1: Consider the Fredholm integral

$$x(t) = 1 + \frac{1}{4}\int_0^1 (s+t)x(s)ds \qquad t \in [0,1].$$

Here$K(t,s) = s + t$, $\lambda = \frac{1}{4}, a = 0\ and\ b = 1$.

Then $|K(t,s)| = |s + t| \le 2 = M$ and $\frac{1}{M(b-a)} = \frac{1}{2} = 0.5$

So that $|\lambda| = \frac{1}{4} < 0.5$ .So this integral equation has unique solution.

Solution: Here $x(t) = 1 + \frac{1}{4}\int_0^1 (t+s)x(s)ds = 1 + \frac{1}{4}tC_1 + \frac{1}{4}C_2$

Where $C_1 = \int_0^1 x(s)ds$ and $C_2 = \int_0^1 sx(s)ds$

Solving these two equations $7C_1 - 2C_2 = 8\ and - 2C_1 + 21C_2 = 12$

And we have $C_1 = \frac{192}{143}$ and $C_2 = \frac{100}{143}$ and solution is

$$x(t) = \frac{24}{143}(7 + 2t).$$

APPLICATION TO VOLTERRA INTEGRAL EQUATION:

Consider the Volterra integral equation:

$$x(t) = f(t) + \lambda \int_a^t K(t,s)x(s)ds \qquad t \in [a,b]$$

Where x(t), K (s, t), f(t) all are belongs to $C[a,b]$.

Let us consider $X = C[a,b]$. the set of all continuous function, and define G-metric on X, where $G(x,y,z) = \begin{cases} 0 & if\ x(t) = y(t) = z(t) \\ 1, & otherwise \end{cases}$

, $t\epsilon[a,b]$.This G-Metric space is also known as Discreate G-metric space and also denoted by $G_{0,1}$.

Theorem 5.1: Let $(X, G_{0,1})$ be a complete metric space where $X = C[a,b]$ and consider the Volterra integral equation:

$$x(t) = f(t) + \lambda \int_a^t K(t,s)x(s)ds \qquad t \in [a,b] \ldots\ldots\ldots\ldots.(v)$$

Where x(t), K (s, t), f(t) all are belongs to $C[a,b]$ and λbe constant.

If $(vi)|K(t,s)| \leq M \quad and\ (vii)|\lambda| < \frac{1}{M(b-a)}$ then Volterra integral equation has unique solution in $C[a,b]$.

Proof: At first, we define an operator $T: C[a,b] \to C[a,b]$ where $X = C[a,b]$ and let $Tx(t) = f(t) + \lambda \int_a^t K(t,s)x(s)ds \quad t \in [a,b]$.

Case-I: Let Tx=Ty=Tz.

Then if $Tx = Ty \Rightarrow f(t) + \lambda \int_a^t K(t,s)x(s)ds = f(t) + \lambda \int_a^t K(t,s)x(s)ds \Rightarrow M \lambda \int_a^t [x(s) - y(s)]ds = 0 \Rightarrow x(s) = y(s) where\ M \neq 0\ and\ \lambda \neq 0$.

Again if $Ty = Tz$ similarly to show that $y(s) = z(s)$.Therefor we have $x(s) = y(s) = z(s) \ldots\ldots\ldots\ldots..(viii)$

Now $G_{0,1}(Tx,Ty,Tz) = 0 = M|\lambda|[\int_a^t |(x(s) - y(s))|ds + \int_a^t |(y(s) - z(s))|ds + \int_a^t |(z(s) - x(s))|ds] = M|\lambda|(b-a).0 = M|\lambda|(b-a)G_{0,1}(x,y,z)$ using (vi) a (viii).

Then by the theorem 3.1 we say that T has a fixed point where $M|\lambda|(b-a). < 1\ i.e\ |\lambda| < \frac{1}{M(b-a)}$,and hence the Given integral equation (v) has a unique solution.

Case -II: Three contractive mapping are not equal i.e. Tx, Ty, Tz are not equal.

Then $|Tx(t) - Ty(t)| = |f(t) + \lambda \int_a^t K(t,s)x(s)ds - f(t) - \lambda \int_a^t K(t,s)y(s)ds| \leq M|\lambda| |\int_a^t |x(s) - y(s)|ds \leq M|\lambda|(b-a)Sup|x(s) - y(s)|$, using (vi)

As if $x \neq y$ then $Sup|x(s) - y(s)| > 0$ and if $M|\lambda|(b-a) < 1\ i.e\ |\lambda| < \frac{1}{M(b-a)}$ Then $Sup(|Tx - Ty)| < Sup(|x(s) - y(s)|$.

Therefor $G_{0,1}(Tx,Ty,Tz) \leq M|\lambda|(b-a)G_{0,1}(x,y,z)$.

Then by the theorem 3.1 we say that T has a fixed point where $M|\lambda|(b-a). < 1\ i.e\ |\lambda| < \frac{1}{M(b-a)}$,and hence the Given integral equation (v) has a unique solution.

Example 5.1: Consider the Volterra equation:

$$x(t) = 1 + \frac{1}{4}\int_0^t (t+s)ds \qquad t \in [0,1]$$

Here $K(t,s) = s + t$, $\lambda = \frac{1}{4}, a = 0 \; and \; b = 1$.

Then $|K(t,s)| = |s+t| \leq 2 = M$ and $\frac{1}{M(b-a)} = \frac{1}{2} = 0.5$

So that $|\lambda| = \frac{1}{4} < 0.5$ .So this integral equation has unique solution.

Conclusion:

In this paper we established the existence and uniqueness results for Fredholm and Volterra integral equation of 2nd kind, using discreate G-metric space. Our results more improve the knowns results.

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