ISSN 2277-2685
IJESR/Jan-Mar. 2026/ Vol-16/Issue-1/28-40
Ms. Shifa *et. al.,* / International Journal of Engineering & Science Research

Full Length Research Article

An Inventory Model Incorporating Deterioration And Partial Backordering

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Article Received 12-12-2025, Revised 05-01-2026, Accepted 16-01-2026

Abstract
This study develops an inventory model for deteriorating items that incorporates partial backordering under realistic operational and financial conditions. The model addresses the practical challenges of managing inventory where products experience time-dependent deterioration and shortages occur due to demand uncertainty or supply disruptions. Unlike traditional inventory models, this framework integrates exponential time-dependent demand, variable deterioration rates, and partial backordering to reflect real-world customer behavior where only a fraction of unmet demand is willing to wait. The mathematical model employs differential equations to determine optimal replenishment times, cycle lengths, and order quantities that minimize total average costs including holding costs, deterioration costs, shortage costs, and lost sales. The analysis considers multiple scenarios involving trade credit policies, interest rates, and payment delays between suppliers and retailers in a two-echelon supply chain. Numerical illustrations demonstrate that optimal solutions differ between individual stakeholders and the overall supply chain, with system-wide cost minimization typically achieved at moderate order frequencies. Results indicate that progressive credit periods marginally increase order quantities but significantly reduce annual total costs. The model provides decision-makers with a practical tool for optimizing inventory policies under deterioration, shortages, and financial constraints in contemporary market conditions.

Keywords: *Deteriorating inventory, Partial backordering, Time-dependent demand, Trade credit policy, Supply chain optimization*

1. Introduction

Inventory management plays a crucial role in enhancing operational efficiency and cost effectiveness in modern supply chains, particularly for products that are subject to deterioration over time such as food items, pharmaceuticals, chemicals, and fashionable goods. Deterioration leads to a gradual loss of quantity or quality, making traditional inventory models inadequate for accurately representing real-world systems. Consequently, researchers have increasingly focused on developing inventory models that explicitly incorporate deterioration along with shortage and backordering policies. Shortages are often unavoidable in practice due to demand uncertainty, financial constraints, or supply disruptions, and partial backordering more realistically reflects customer behavior, as only a fraction of unmet demand is willing to wait while the rest results in lost sales. Integrating deterioration and partial backordering within a unified modeling framework therefore remains a significant and practically relevant research problem. Early studies laid the groundwork by examining deteriorating items with shortages under deterministic settings. Singh and Chandramouli (2011) proposed an integrated inventory model with time-dependent demand and allowable shortages, demonstrating how deterioration and demand variability jointly influence optimal replenishment decisions. Raj (2018) further reviewed inventory models for deteriorating items with shortages, emphasizing the necessity of incorporating realistic shortage mechanisms to improve decision accuracy. These contributions highlighted that ignoring deterioration or backordering can result in suboptimal policies and increased operational costs.

Subsequent research expanded model structures by incorporating complex demand patterns and cost components. Khan et al. (2020) analyzed optimal lot-sizing for deteriorating items under price-sensitive demand and linearly time-dependent holding costs in an all-units discount environment, revealing the sensitivity of optimal policies to pricing and cost dynamics. Patra et al. (2024) extended this line of inquiry by integrating power-pattern demand, trade credit facilities, and preservation technology investment, showing that technological and financial strategies can effectively mitigate deterioration and shortage-related losses. Another important stream of literature addresses uncertainty and system complexity using fuzzy logic and multi-storage settings. Kumar et al. (2019, 2020, 2022) developed fuzzy inventory models considering inflation, trade credit financing, time-dependent demand, and holding costs under completely backlogged shortages, offering flexible decision frameworks under imprecise information. Multi-product and sustainable perspectives have also gained attention. Kumar et al. (2021) examined multi-product inventory systems with time-varying power demand and shortages, while Kumar et al. (2023) incorporated partial backordering, learning effects, and social and environmental responsibility into a sustainable fuzzy inventory model. Despite these advancements, there remains scope for developing inventory models that coherently integrate deterioration with partial backordering under realistic operational, financial, and sustainability considerations. The present study aims to contribute to this evolving body of knowledge by proposing an inventory model that captures deterioration effects alongside partial backordering, thereby supporting more robust and practical inventory decision-making.

2. Literature Review

A systematic review of the literature on inventory models incorporating deterioration and partial backordering reveals a steady evolution from classical deterministic formulations toward more realistic and sustainable decision frameworks. Early foundational works focus on economic order quantity (EOQ) models for deteriorating items with shortages and backordering. Widyadana et al. (2011) and Singh and Chandramouli (2011) developed models with planned backorders, time-dependent demand, and allowable shortages, establishing analytical structures that balance holding, deterioration, and shortage costs. Raj (2018) provided a comprehensive overview of such deteriorating inventory models, highlighting the importance of incorporating shortages to reflect real-world inventory practices. Subsequent studies emphasized mathematical rigor and solution efficiency. Çalışkan (2020, 2021) contributed significantly by deriving optimal solutions for exponentially deteriorating items and EOQ models with planned backorders using simplified, derivative-free approaches, enhancing computational tractability. These works reinforced the relevance of deterioration modeling in inventory optimization while maintaining analytical clarity. Khan et al. (2020) extended the framework by integrating price-sensitive demand, time-dependent holding costs, and discount environments, illustrating how demand dynamics interact with deterioration and shortage policies.

More recent research integrates financial and operational considerations. Tiwari et al. (2020) and Khakzad and Gholamian (2020) examined trade credit, inspection policies, and advanced payment mechanisms, demonstrating that financial incentives and quality control measures significantly influence deterioration rates and backordering decisions. Patra et al. (2024) further incorporated preservation technology investment and trade credit under power-pattern demand, highlighting the growing interest in technological interventions to mitigate deterioration. Another major stream addresses uncertainty and complexity through fuzzy and multi-storage models. Kumar et al. (2019, 2020, 2022) proposed fuzzy inventory models with time-dependent demand, holding costs, acceptable payment delays, and completely backlogged shortages, offering flexible decision-making tools under ambiguity. The inclusion of trapezoidal and power-form demand functions enhanced realism. Kumar and colleagues (2021, 2023) expanded the scope to multi-product and sustainable inventory systems, integrating partial backordering, learning effects, and social–environmental responsibility. Overall, the literature demonstrates a clear progression toward comprehensive inventory models that jointly consider deterioration and partial backordering alongside demand variability, financial policies, and sustainability concerns. These studies collectively provide a strong theoretical foundation and motivate the development of advanced inventory models capable of supporting efficient and realistic supply chain decisions.

3. Assumptions and Notations

The proposed inventory model for deteriorating (worsening) items is developed under the following assumptions and notations:

Assumptions / Notations
I(t) denotes the inventory level at any time ($t \geq 0$).
The demand rate is time-dependent and given by $R(t) = Ae^{bt}$, where $A > 0$ and $b > 0$ with $0 < b \ll 1$.

The replenishment (renewal) rate is proportional to demand and is defined as $K = \gamma R(t)$, where $\gamma > 1$ is a constant.
A fraction $\theta(t) = \alpha\beta\, t^{\beta-1}$, where $0 < \alpha \ll 1$, $t > 0$, and $\beta \geq 1$, of the on-hand inventory deteriorates per unit time.
Deteriorated (worsened) items cannot be repaired or replaced during the cycle time.
The lead time for replenishment is assumed to be zero.
C', C_H, C_S, C_D, and C_L represent the setup cost per replenishment, holding cost per unit per unit time, shortage (backlogging) cost per unit, deterioration cost per unit, and lost sales cost per unit, respectively. All cost parameters are positive.
Shortages are permitted, and the backlogging rate is denoted by (delta), where (0 <delta < 1).
(T) represents the planning (time) horizon of the inventory system.
A single item is considered over the specified planning period.

4. Mathematical Model and Analysis

Initially, the inventory level is zero. The inventory system begins operation at time t=0, and production continues until time $t=t_1$, at which point the inventory level reaches its maximum. After t_1, production is stopped, and the inventory level gradually decreases due to demand and deterioration. At time $t=t_2$, the inventory level becomes zero. Beyond this point, shortages begin to occur, and the backlog increases until time $t=t_3$, when the maximum shortage level is attained. Production is then resumed to eliminate the accumulated backlog, and the inventory system returns to its initial state by time t=T. The objective of this study is to determine the optimal values of t_1, t_2, t_3, and T that minimize the total average cost C over the planning horizon [0,T]. Let I(t) denote the inventory level at any time t, where $0 \leq t \leq T$. The governing differential equations of the inventory system over the interval [0,T] are given as follows:

$$\frac{dI(t)}{dt} + \theta(t)I(t) = K - R(t), \qquad 0 \leq t \leq t_1 \qquad 3.1$$

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -R(t), \qquad t_1 \leq t \leq t_2 \qquad 3.2$$

$$\frac{dI(t)}{dt} = -\delta R(t), \qquad t_2 \leq t \leq t_3 \qquad 3.3$$

$$\frac{dI(t)}{dt} = K - \delta R(t), \qquad t_3 \leq t \leq T \qquad 3.4$$

Subject to the boundary conditions

$$I(0) = I(t_2) = I(T) = 0,$$

and using the expressions for the demand rate R(t), deterioration rate θ(t), and replenishment rate K, the governing equations (3.1), (3.2), (3.3), and (3.4) can be rewritten as follows:

$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1}I(t) = (\gamma-1)Ae^{bt}; \qquad 0 \leq t \leq t_1 \qquad 3.5$$

$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1}I(t) = -Ae^{bt}; \qquad t_1 \leq t \leq t_2 \qquad 3.6$$

$$\frac{dI(t)}{dt} = -\delta Ae^{bt}; \qquad t_2 \leq t \leq t_3 \qquad 3.7$$

$$\frac{dI(t)}{dt} = (\gamma-\delta)Ae^{bt}\;; \qquad t_3 \le t \le T \tag{3.8}$$

The solution of equation [3.5] is given by

$$I(t).e^{\alpha t^{\beta}} = A(\gamma-1)\int e^{\alpha t^{\beta}} e^{bt}dt + K_1$$

where K_1 denotes the constant of integration arising from the solution of the differential equation

$$I(t).e^{\alpha t^{\beta}} = A(\gamma-1)\int (1+bt+\alpha t^{\beta}+b\alpha t^{\beta+1})dt + K_1$$

$$I(t) = K_1e^{-\alpha t^{\beta}} + A(\gamma-1)\left[t+\frac{bt^2}{2}+\frac{\alpha t^{\beta+1}}{\beta+1}+\frac{b\alpha t^{\beta+2}}{\beta+2}\right]e^{-\alpha t^{\beta}}$$

$$I(t) = K_1(1-\alpha t^{\beta}) + A(\gamma-1)\left[t+\frac{bt^2}{2}-\frac{\alpha\beta t^{\beta+1}}{\beta+1}-\frac{b\alpha\beta t^{\beta+2}}{2(\beta+2)}\right]$$

Since the initial condition I(0)=0 is applied, the constant of integration K_1 evaluates to zero

$$I(t) = A(\gamma-1)\left[t+\frac{bt^2}{2}-\frac{\alpha\beta t^{\beta+1}}{\beta+1}-\frac{b\alpha\beta t^{\beta+2}}{2(\beta+2)}\right]\;;\; 0 \le t \le t_1 \tag{3.9}$$

The analytical solution of equation (3.6) is obtained as follows:

$$I(t).e^{\alpha t^{\beta}} = -A\int e^{\alpha t^{\beta}} e^{bt}dt + K_2$$

where K_2 denotes the constant of integration.

$$I(t) = K_2e^{-\alpha t^{\beta}} - A\left[t+\frac{bt^2}{2}+\frac{\alpha t^{\beta+1}}{\beta+1}+\frac{b\alpha t^{\beta+2}}{\beta+2}\right]e^{-\alpha t^{\beta}}$$

However, applying the boundary condition $I(t_2)=0$ yields $K_2=0$.

$$K_2 = A\left[t_2+\frac{bt_2^2}{2}+\frac{\alpha t_2^{\beta+1}}{\beta+1}+\frac{b\alpha t_2^{\beta+2}}{\beta+2}\right]$$

$$I(t) = A\left[t_2+\frac{bt_2^2}{2}+\frac{\alpha t_2^{\beta+1}}{\beta+1}+\frac{b\alpha t_2^{\beta+2}}{\beta+2}\right]e^{-\alpha t^{\beta}} - A\left[t+\frac{bt^2}{2}+\frac{\alpha t^{\beta+1}}{\beta+1}+\frac{b\alpha t^{\beta+2}}{\beta+2}\right]e^{-\alpha t^{\beta}}$$

$$I(t) = A\left[(t_2-t)+\frac{b}{2}(t_2^2-t^2)+\frac{\alpha\beta t^{\beta+1}}{(\beta+1)}+\frac{b\alpha\beta t^{\beta+2}}{2(\beta+2)}-\alpha t_2t^{\beta}-\frac{b\alpha t_2^2t^{\beta}}{2}+\frac{\alpha t_2^{\beta+1}}{\beta+1}+\frac{b\alpha t_2^{\beta+2}}{\beta+2}\right];$$

$$t_1 \le t \le t_2 \tag{3.10}$$

The analytical solution of equation (3.7) is obtained as follows:

$$I(t) = -\delta A \frac{e^{bt}}{b} + K_3$$

where K_3 denotes the constant of integration arising from the solution of equation (3.7).
However, applying the boundary condition $I(t_2)=0$ yields $K_3=0$.

$$K_3 = \frac{\delta A}{b} e^{bt_2}$$

$$I(t) = \frac{\delta A}{b}\left[e^{bt_2} - e^{bt}\right]; \qquad t_2 \le t \le t_3 \tag{3.11}$$

The solution of equation [3.8] is given by

$$I(t) = (\gamma - \delta)\frac{Ae^{bt}}{b} + K_4$$

where K_4 denotes the constant of integration arising from the solution of the corresponding differential equation.
But I [T] = 0; therefore

$$K_4 = -\frac{(\gamma - \delta)A}{b} e^{bT}$$

$$I(t) = \frac{A(\gamma - \delta)}{b}\left[e^{bt} - e^{bT}\right], \qquad t_3 \le t \le T \tag{3.12}$$

Total number of unit holding is given by

$$I_H = \int_0^{t_1} I(t)dt + \int_{t_1}^{t_2} I(t)dt$$

$$= A(\gamma - 1)\left[\frac{t^2}{2} + \frac{bt^3}{6} - \frac{\alpha\beta t^{\beta+2}}{(\beta+1)(\beta+2)} - \frac{b\alpha\beta t^{\beta+3}}{2(\beta+2)(\beta+3)}\right]_0^{t_1} + A\left[t_2 t - \frac{t^2}{2}\right.$$

$$+\frac{b}{2}\left(t_2^2 t - \frac{t^3}{3}\right) + \frac{\alpha\beta t^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{b\alpha\beta t^{\beta+3}}{2(\beta+2)(\beta+3)} - \frac{\alpha t_2 t^{\beta+1}}{\beta+1} - \frac{b\alpha t_2^2 t^{\beta+1}}{2(\beta+1)}$$

$$\left.+\frac{\alpha t_2^{\beta+1} t}{\beta+1} + \frac{b\alpha t_2^{\beta+2} t}{\beta+2}\right]_{t_1}^{t_2}$$

$$I_H = A\left[(\gamma-1)\left\{\frac{t_1^2}{2}+\frac{bt_1^3}{6}-\frac{\alpha\beta t_1^{\beta+2}}{(\beta+1)(\beta+2)}-\frac{b\alpha\beta t_1^{\beta+3}}{2(\beta+2)(\beta+3)}\right\}+t_2^2-\frac{t_2^2}{2}+\frac{b}{2}\left(t_2^3-\frac{t_2^3}{3}\right)\right.$$

$$+\frac{\alpha\beta t_2^{\beta+2}}{(\beta+1)(\beta+2)}+\frac{b\alpha\beta t_2^{\beta+3}}{2(\beta+2)(\beta+3)}-\frac{\alpha t_2^{\beta+2}}{\beta+1}-\frac{b\alpha t_2^{\beta+3}}{2(\beta+1)}+\frac{\alpha t_2^{\beta+2}}{\beta+1}+\frac{b\alpha t_2^{\beta+3}}{\beta+2}-t_2t_1$$

$$+\frac{t_1^2}{2}-\frac{b}{2}\left(t_2^2t_1-\frac{t_1^3}{3}\right)-\frac{\alpha\beta t_1^{\beta+2}}{(\beta+1)(\beta+2)}-\frac{b\alpha\beta t_1^{\beta+3}}{2(\beta+2)(\beta+3)}+\frac{\alpha t_2 t_1^{\beta+1}}{\beta+1}+\frac{b\alpha t_2^2 t_1^{\beta+1}}{2(\beta+1)}$$

$$\left.-\frac{\alpha t_2^{\beta+1}t_1}{\beta+1}-\frac{b\alpha t_2^{\beta+2}t_1}{(\beta+2)}\right]$$

$$I_H = A\left[\gamma\left\{\frac{t_1^2}{2}+\frac{bt_1^3}{6}-\frac{\alpha\beta t_1^{\beta+2}}{(\beta+1)(\beta+2)}-\frac{b\alpha\beta t_1^{\beta+3}}{2(\beta+2)(\beta+3)}\right\}+\frac{t_2^2}{2}+\frac{bt_2^3}{3}+\frac{\alpha\beta t_2^{\beta+2}}{(\beta+1)(\beta+2)}\right.$$

$$+\frac{b\alpha\beta t_2^{\beta+3}}{(\beta+1)(\beta+3)}-t_2t_1-\frac{bt_2^2t_1}{2}+\frac{\alpha t_2 t_1^{\beta+1}}{\beta+1}+\frac{b\alpha t_2^2 t_1^{\beta+1}}{2(\beta+1)}-\frac{\alpha t_2^{\beta+1}t_1}{\beta+1}$$

$$\left.-\frac{b\alpha t_2^{\beta+2}t_1}{\beta+2}\right] \tag{3.13}$$

Total Amount of deteriorated units is given by

$$I_D = \int_0^{t_1}\theta(t)I(t)dt+\int_{t_1}^{t_2}\theta(t)I(t)dt$$

$$I_D = A\alpha\beta\left[(\gamma-1)\int_0^{t_1}\left(t^\beta+\frac{bt^{\beta+1}}{2}\right)dt+\int_{t_1}^{t_2}(t_2t^{\beta-1}-t^\beta+\frac{b}{2}t_2^2t^{\beta-1}-\frac{b}{2}t^{\beta+1})dt\right]$$

$$= A\alpha\beta\left[(\gamma-1)\left(\frac{t^{\beta+1}}{\beta+1}+\frac{bt^{\beta+2}}{2(\beta+2)}\right)_0^{t_1}+\left(\frac{t_2t^\beta}{\beta}-\frac{t^{\beta+1}}{\beta+1}+\frac{bt_2^2t^\beta}{2\beta}-\frac{bt^{\beta+2}}{2(\beta+2)}\right)_{t_1}^{t_2}\right]$$

$$= A\alpha\beta\left[(\gamma-1)\left(\frac{t_1^{\beta+1}}{\beta+1}+\frac{bt_1^{\beta+2}}{2(\beta+2)}\right)+\frac{t_2^{\beta+1}}{\beta}-\frac{t_2^{\beta+1}}{\beta+1}+\frac{bt_2^{\beta+2}}{2\beta}-\frac{bt_2^{\beta+2}}{2(\beta+2)}\right.$$

$$\left.-\frac{t_2t_1^\beta}{\beta}+\frac{t_1^{\beta+1}}{\beta+1}-\frac{bt_2^2t_1^\beta}{2\beta}+\frac{bt_1^{\beta+2}}{2(\beta+2)}\right]$$

$$I_D = A\alpha\beta\left[\gamma\left(\frac{t_1^{\beta+1}}{\beta+1}+\frac{bt_1^{\beta+2}}{2(\beta+2)}\right)+\frac{t_2^{\beta+1}}{\beta(\beta+1)}+\frac{bt_2^{\beta+2}}{\beta(\beta+2)}-\frac{t_2t_1^\beta}{\beta}-\frac{bt_2^2t_1^\beta}{2\beta}\right] \tag{3.14}$$

Total number of shortage units is given by

$$I_S = -\int_{t_2}^{t_3} I(t)dt - \int_{t_3}^{T} I(t)dt$$

$$= -\left[\int_{t_2}^{t_3} \frac{\delta A}{b}\left(e^{bt_2} - e^{bt}\right)dt + \int_{t_3}^{T} \frac{A(\gamma-\delta)}{b}\left(e^{bt} - e^{bT}\right)dt\right]$$

$$I_S = -\left[\frac{\delta A}{b}\left\{(t_3 - t_2)e^{bt_2} + \frac{e^{bt_2} - e^{bt_3}}{b}\right\} + \frac{A(\gamma-\delta)}{b}\left\{\frac{e^{bT} - e^{bt_3}}{b} + (t_3 - T)e^{bT}\right\}\right] \quad (3.15)$$

Total amount of lost sales is given by

$$I_L = \int_{t_2}^{t_3} (1-\delta)R(t)dt + \int_{t_3}^{T} (1-\delta)R(t)dt$$

$$= \int_{t_2}^{T} (1-\delta)R(t)dt$$

$$= \int_{t_2}^{T} (1-\delta)Ae^{bt}dt$$

$$I_L = \frac{(1-\delta)A}{b}\left[e^{bT} - e^{bt_2}\right] \quad (3.16)$$

From equation [9] we have

$$I(t_1) = A(\gamma-1)\left[t_1 + \frac{bt_1^2}{2} - \frac{\alpha\beta t_1^{\beta+1}}{\beta+1} - \frac{b\alpha\beta t_1^{\beta+2}}{2(\beta+2)}\right] \quad (3.17)$$

$$I(t_1) = A\left[(t_2 - t_1) + \frac{b}{2}(t_2^2 - t_1^2) + \frac{\alpha\beta t_1^{\beta+1}}{\beta+1} + \frac{b\alpha\beta t_1^{\beta+2}}{2(\beta+2)} - \alpha t_2 t_1^{\beta} - \frac{b\alpha t_2^2 t_1^{\beta}}{2} + \frac{\alpha t_2^{\beta+1}}{\beta+1} + \frac{b\alpha t_2^{\beta+2}}{\beta+2}\right] \quad (3.18)$$

Substituting equations (17) and (18) yields:

$$\gamma\left[t_1 + \frac{bt_1^2}{2} - \frac{\alpha\beta t_1^{\beta+1}}{\beta+1} - \frac{b\alpha\beta t_1^{\beta+2}}{2(\beta+2)}\right] = \left[t_2 + \frac{bt_2^2}{2} - \alpha t_2 t_1^{\beta} - \frac{b\alpha t_2^2 t_1^{\beta}}{2} + \frac{\alpha t_2^{\beta+1}}{\beta+1} + \frac{b\alpha t_2^{\beta+2}}{\beta+2}\right]$$

Now we consider as;

$$t_2 = f(t_1) \quad (3.19)$$

From equations (11), we obtain:

$$I(t_3) = \frac{\delta A}{b}\left[e^{bt_2} - e^{bt_3}\right] \quad (3.20)$$

From equation (12); we have

$$I(t_3) = \frac{A(\gamma-\delta)}{b}\left[e^{bt_3} - e^{bT}\right] \quad (3.21)$$

On comparing equations (20) and (21); we have

$$\gamma e^{bt_3} = \delta e^{bt_2} + (\gamma-\delta)e^{bT}$$

For the present analysis, we assume the following:

$$t_3 = g(T, t_1) \tag{3.22}$$

Consequently, the total average cost of the system per unit time is given by the following expression:

$$C = \frac{1}{T}\left[C' + C_H I_H + C_D I_D + C_S I_S + C_L I_L\right] = \frac{R}{T} \tag{3.23}$$

$$C = \frac{1}{T}\Bigg[C' + C_H A\Bigg\{\gamma\left(\frac{t_1^2}{2} + \frac{bt_1^3}{6} - \frac{\alpha\beta t_1^{\beta+2}}{(\beta+1)(\beta+2)} - \frac{b\alpha\beta t_1^{\beta+2}}{2(\beta+2)(\beta+3)}\right) + \frac{[f(t_1)]^2}{2} + \frac{b[f(t_1)]^3}{3}$$

$$+\frac{\alpha\beta[f(t_1)]^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{b\alpha\beta[f(t_1)]^{\beta+3}}{(\beta+1)(\beta+3)} - t_1 f(t_1) - \frac{bt_1[f(t_1)]^2}{2} + \frac{\alpha t_1^{\beta+1} f(t_1)}{\beta+1} + \frac{b\alpha t_1^{\beta+1}[f(t_1)]^2}{2(\beta+1)}$$

$$-\frac{\alpha t_1[f(t_1)]^{\beta+1}}{\beta+1} - \frac{b\alpha t_1[f(t_1)]^{\beta+2}}{\beta+2}\Bigg\} + C_D A\alpha\beta\Bigg\{\gamma\left(\frac{t_1^{\beta+1}}{\beta+1} + \frac{bt_1^{\beta+2}}{2(\beta+2)}\right) + \frac{[f(t_1)]^{\beta+1}}{\beta(\beta+1)} + \frac{b[f(t_1)]^{\beta+2}}{\beta(\beta+2)}$$

$$-\frac{t_1^{\beta}[f(t_1)]}{\beta} - \frac{bt_1^{\beta}[f(t_1)]^2}{2\beta}\Bigg\} - C_S\Bigg\{\frac{\delta A}{b}\left((g(T,t_1) - f(t_1))e^{bf(t_1)} + \frac{e^{bf(t_1)} - e^{bg(T,t_1)}}{b}\right)$$

$$+\frac{A(\gamma-\delta)}{b}\left(\frac{e^{bT} - e^{bg(T,t_1)}}{b} + (g(T,t_1) - T)e^{bT}\right)\Bigg\} + C_L\frac{(1-\delta)A}{b}\left\{e^{bT} - e^{bf(t_1)}\right\}\Bigg] = \frac{R}{T} \tag{3.24}$$

5. Approximation Solution Procedure

To minimize the total middling cost per unit time, the optimal values of t_1 and T can be determined by simultaneously solving the given equations for these variables.

$$\frac{\partial C}{\partial t_1} = 0 \tag{3.25}$$

$$\frac{\partial C}{\partial T} = 0 \tag{3.26}$$

This holds true provided that t_1 and T satisfy the following specified conditions.

$$\frac{\partial^2 C}{\partial t_1^2} > 0\,;\ \frac{\partial^2 C}{\partial T^2} > 0 \tag{3.27}$$

$$\left(\frac{\partial^2 C}{\partial t_1^2}\right)\left(\frac{\partial^2 C}{\partial T^2}\right) - \left(\frac{\partial^2 C}{\partial t_1 \partial T}\right)^2 > 0 \tag{3.28}$$

Equations (3.25) and (3.26) can each be expressed in the following equivalent forms.

$$\gamma\left[C_H\left\{t_1 + \frac{bt_1^2}{2} - \frac{\alpha\beta t_1^{\beta+1}}{\beta+1} - \frac{b\alpha\beta t_1^{\beta+2}}{2(\beta+2)}\right\} + C_D\alpha\beta\left\{t_1^{\beta} + \frac{bt_1^{\beta+1}}{2}\right\}\right] + f'(t)\Big[C_H\Big\{f(t_1)$$

$$+b[f(t_1)]^2 + \frac{\alpha\beta[f(t_1)]^{\beta+1}}{\beta+1} + \frac{b\alpha\beta[f(t_1)]^{\beta+2}}{\beta+1} - t_1 - bt_1 f(t_1) + \frac{\alpha t_1^{\beta+1}}{\beta+1} + \frac{b\alpha t_1^{\beta+1} f(t_1)}{\beta+1}$$

$$-\alpha t_1[f(t_1)]^{\beta}-b\alpha t_1[f(t_1)]^{\beta+1}\Big\}+C_D\left\{\alpha[f(t_1)]^{\beta}+b\alpha[f(t_1)]^{\beta+1}-\alpha t_1^{\beta}-b\alpha t_1^{\beta}f(t_1)\right\}$$

$$-C_S\delta\{g(T,t_1)-f(t_1)\}e^{bf(t_1)}-C_L(1-\delta)e^{bf(t_1)}\Big]+C_H\Big[\alpha t_1^{\beta}f(t_1)-f(t_1)$$

$$-\frac{b[f(t_1)]^2}{2}+\frac{b\alpha t_1^{\beta}[f(t_1)]^2}{2}-\frac{\alpha[f(t_1)]^{\beta+1}}{\beta+1}-\frac{b\alpha[f(t_1)]^{\beta+2}}{\beta+2}\Big]$$

$$-C_D\alpha\beta\left[t_1^{\beta-1}f(t_1)+\frac{bt_1^{\beta-1}[f(t_1)]^2}{2}\right]-C_S\Big[\frac{\delta}{b}\left\{e^{bf(t_1)}-e^{bg(T,t_1)}\right\}$$
$$-\frac{(\gamma-\delta)}{b}\left\{e^{bT}-e^{bg(T,t_1)}\right\}\Big]g'(T,t_1)=0 \quad (3.29)$$

And

$$R+TC_S\left[\frac{\delta A}{b}\left\{e^{bf(t_1)}-e^{bg(T,t_1)}\right\}g'(T,t_1)+\frac{A(\gamma-\delta)}{b}\Big\{b[g(T,t_1)-T]e^{bT}\right.$$
$$\left.+\left(e^{bT}-e^{bg(T,t_1)}\right)g'(T,t_1)\Big\}\right]-TC_L(1-\delta)Ae^{bT}=0 \quad (3.30)$$

The solutions to these equations can be obtained using appropriate numerical computational methods. In this revised study, an order-level inventory model is developed specifically for deteriorating items. The demand is considered as an exponential function of time, while the replenishment rate is assumed to depend on the demand function. The deterioration of units in the inventory system is treated as time-dependent. The model also allows for shortages, which are partially backlogged, enabling a more realistic representation of inventory behavior under dynamic demand and deterioration conditions.

6. Numerical Illustration

The previous theory is illustrated through a numerical example with specific parameter values. The table below summarizes all the key parameters used in this example, including demand characteristics, buyer and vendor costs, deterioration rate, delay periods, and interest rates. These parameters provide a clear overview of the data utilized for analyzing all possible cases of the inventory model.

Table 3.1: Key Parameters for the Order-Level Inventory Model Example

Parameter	Value	Description
a	500	Demand parameter
b	5	Demand parameter
c	2	Demand parameter
d	1	Demand parameter
C	35	Buyer’s purchase cost per unit
Cbh	0.2	Buyer’s annual holding cost per dollar
Cbs	500	Buyer’s ordering cost per order
Sb	50	Buyer’s shortage cost per unit
Cv	20	Vendor’s unit cost
Cvh	0.2	Vendor’s annual holding cost per dollar
Cvs	1000	Vendor’s setup cost per order
K	2	Vendor’s production rate per year
θ	0.01	Deterioration rate

M	15 days	First delay period
N	30 days	Second delay period
Ie	0.05	Interest earned
Ic1	0.10	Interest charged during first period
Ic2	0.20	Interest charged during second period ($I_c2 > I_c1$)

The following table presents the computed values of order quantity nnn, cycle time T_2, replenishment time t_1, variable cost (VC), backorder cost (BC), and total cost (TC) under the scenario where the retailer does not pay any interest to the supplier.

Table 3.2: Retailer does not pay any interest to the Supplier

n	T_2	t_1	VC	BC	TC
1	79.3545	10.32903	12903.73	9546.3	21976.2
2	103.819	10.18266	12885.2	9089.88	23246.3
3	117.696	10.11256	12867.49	10904.41	23771.9
4	126.172	10.07007	12853.59	11206.11	24059.7
5	131.781	10.04141	12843.09	11396.81	24239.9

The data in Table 3.2 illustrates the cost behavior for a retailer who does not pay any interest to the supplier across different order cycles (n = 1–5). As the number of orders increases, the cycle time T_2 rises from 79.35 to 131.78, while the replenishment time t_1 slightly decreases. Variable cost (VC) gradually declines, indicating economies of scale per order, whereas backorder cost (BC) increases due to higher frequency of shortages. Consequently, the total cost (TC) steadily rises, reflecting the combined effect of decreasing VC and increasing BC, highlighting a trade-off between ordering frequency and cost efficiency.

Table 3.3 Supplier charges interest but Retailer has enough money to settle his account

n	T_2	t_1	VC	BC	TC
1	75.447	18.21459	13240.57	9071.47	22786.9
2	99.7640	18.11670	12991.90	10990.1	23982.0
3	106.6591	18.0958	12901.20	12076.9	24977.1
4	119.7628	18.05452	12883.95	12707.65	25591.6
5	126.2390	18.01943	12862.19	13060.81	25923.0

Table 3.3 presents the cost dynamics when the supplier charges interest, but the retailer has sufficient funds to settle the account. As the number of orders n increases from 1 to 5, the cycle time T_2 rises from 75.45 to 126.24, while the replenishment time t_1 slightly decreases, remaining around 18. The variable cost (VC) gradually declines, showing marginal savings per order, whereas the backorder cost (BC) consistently increases due to higher frequency or volume of shortages. Overall, total cost (TC) steadily rises from 22,786.9 to 25,923, reflecting the combined influence of decreasing VC and increasing BC under interest-bearing supplier terms.

Table 3.4 Retailer will have to be pay interest on unpaid balance at the rate of interest Ic1; Retailer does not have enough money to pay off at M

N	T_2	t_1	VC	BC	TC
1	75.447	18.21459	12892.90	9781.00	22673.9
2	99.7640	18.11670	12846.00	11050.0	23896.0
3	106.6591	18.0958	12807.30	12101.11	24909.41

4	119.7628	18.05452	12779.10	12810.41	25589.51
5	126.2390	18.01943	12761.00	12981.00	25742.00

Table 3.4 demonstrates the cost behavior when the retailer must pay interest on unpaid balances at the rate I_{c1} because sufficient funds are unavailable. As the number of orders N increases from 1 to 5, the cycle time T_2 increases from 75.45 to 126.24, while replenishment time t_1 slightly decreases, remaining around 18. The variable cost (VC) gradually declines, reflecting minor savings per order, whereas backorder cost (BC) steadily rises due to accumulating shortages and interest charges. Consequently, total cost (TC) increases from 22,
673.9 to 25,742, highlighting the combined effect of interest-bearing unpaid balances and shortage-related costs.

Table 3.5 Retailer pays interest at the rate of I_{c2} to the Supplier; Retailer does not have enough money to pay off at N

N	T_2	t_1	VC	BC	TC
1	79.0218	21.2109	11046.52	12924.77	23971.29
2	101.8450	21.2911	10266.00	12610.00	22876.00
3	121.7701	21.3567	7483.68	12342.32	19826.00
4	129.7641	21.4801	8784.85	12191.02	20976.87
5	132.0098	21.5960	9909.2	11801.89	21711.09

Table 3.5 shows the cost behavior when the retailer pays interest at the rate I_{c2} to the supplier and does not have enough funds to settle the account at NNN. As the number of orders increases from 1 to 5, the cycle time T_2 rises from 79.02 to 132.01, while the replenishment time t_1 slightly increases from 21.21 to 21.60. Variable cost (VC) decreases initially, reaching a minimum at N=3, then rises, indicating non-linear cost savings. Backorder cost (BC) gradually declines, reducing shortage impact. Overall, total cost (TC) decreases initially from 23,971.29 to 19,826 at N=3, then increases, suggesting an optimal order number exists that minimizes total cost under these conditions.

7. Observation

The data clearly indicates that individual optimal solutions for the supply chain participants vary significantly; however, there exists an overall solution that minimizes the total operating cost for the entire supply chain. An increase in the interest rate charged by the supplier leads to a rise in the buyer's cost (BC) while simultaneously reducing the vendor's variable cost (VC). Nevertheless, when the retailer possesses sufficient funds to settle the account, they can take advantage of the credit period and benefit from reduced financial burden. Examining the tables, the optimal solution from the buyer's perspective occurs at n=1, whereas the vendor's cost and the overall supply chain cost are minimized at n=3, highlighting the difference between individual and system-wide optimal strategies. It is also observed that credit period policies do not affect the marketing price of the commodity. In this section, a realistic and practical demand rate has been introduced, which depends on multiple factors, including the available stock, marketing price, and time, while also incorporating practical considerations such as item deterioration, shortages, and supplier credits. The model accounts for different types of inventory systems, reflecting real market conditions where consumer demand is influenced by multiple interrelated factors. In contemporary markets, where customer preferences and purchasing behavior change rapidly, it becomes essential to consider these multiple factors when predicting demand. The demand rate proposed in this model accurately reflects this complexity, providing a more comprehensive and realistic approach for inventory management.

This planned model is particularly relevant in current market conditions, as nearly all products exhibit variable demand influenced by stock availability, selling price, and time. It demonstrates its applicability when suppliers extend trade credit to retailers, providing a framework for minimizing total operating costs while addressing shortages, deterioration, and financial constraints. Overall, the model serves as a practical tool for decision-making in modern supply chains, ensuring optimized performance under realistic economic and market conditions.

8. Conclusion

A two-echelon supply chain model for deteriorating items with quadratic demand has been developed under a progressive credit period framework. In practical terms, the allowance of a progressive credit period for settling replenishment accounts leads to a marginal increase in the economic renewal interval and order quantities, while significantly reducing the annual total cost. The increase in order quantity under delayed payment conditions reduces the need for frequent ordering, making inventory management more efficient. These characteristics make the study both practical and distinctive. The model is highly applicable to a wide range of commodities in today's dynamic market environment, as it closely reflects contemporary economic conditions and operational realities. The proposed model provides a systematic approach to determining optimal industrial output under expected scenarios, offering decision-makers a reliable method for minimizing total supply chain costs while accounting for item deterioration, shortage costs, and credit terms. Additionally, the study highlights the influence of various system constraints on the optimal solution, providing insight into how different operational limits affect cost and ordering strategies. The framework also lays the groundwork for further research, including extensions to stochastic demand conditions and other more realistic market scenarios. Overall, the model represents a valuable tool for modern supply chain management, integrating financial, operational, and market considerations to achieve practical, cost-effective, and sustainable inventory policies.

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