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A Fixed Point Theorem for $\in -$Chainable Intuitionistic Fuzzy Metric Space

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Abstract

The field of intuitionistic fuzzy fixed point theory has attracted considerable attention from researchers, as the development of intuitionistic fuzzy mathematics has introduced novel directions for exploration in fixed point analysis. In this paper, we establish certain conditions under which four self−mappings on a chainable intuitionistic fuzzy metric space possess a unique common fixed point.

Keywords

Fixed Points, Fuzzy Sets, Fuzzy Metric Spaces, Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy Metric Spaces.

1 Introduction

Fuzzy mathematics began with the pioneering work of Zadeh [12] who introduced the concept of fuzzy sets to model imprecise and uncertain information encountered in real−world scenarios. In mathematical programming, many problems aim to optimize objective functions under given constraints and in practical applications, these often involve multiple, sometimes conflicting, goals. Finding a solution that simultaneously optimizes all objective functions can be extremely challenging. A notable approach to address this complexity involves the application of fuzzy sets, as proposed by D. Turkoglu and B. E. Rhoades [11]. Expanding on this idea, Atanassov [4] introduced intuitionistic fuzzy sets, which extended fuzzy set theory by allowing both membership and non−membership degrees, thereby offering a more flexible representation of uncertainty. In this framework, membership is not represented by a single value but rather by a pair of values indicating the degree of membership and non-membership, with the requirement that their sum does not exceed one. Unlike classical fuzzy sets where the degree of non−membership is simply the complement of the membership degree intuitionistic fuzzy sets allow for hesitation, making them better suited for situations involving ambiguity and partial knowledge. Many properties and results established for fuzzy sets can be adapted to intuitionistic fuzzy sets, although the converse does not always hold. The additional degree of freedom provided by the hesitation margin makes intuitionistic fuzzy sets a more refined tool for dealing with uncertainty in various contexts. Several researchers, including Coker [5],

S. Sharma [10], Gregori and Sapena [6] and C. Alaca et al. [1,2,3] have contributed to the study of fixed point and common fixed point theorems within fuzzy and intuitionistic fuzzy metric spaces. These studies often rely on contractive−type conditions to establish the existence of such points. Saadati and Park [9] explored intuitionistic fuzzy metric spaces and introduced the concept of Cauchy sequences in this setting. They extended classical fixed point results, such as Banach's and Edelstein's theorems to intuitionistic fuzzy metric spaces. Furthermore, Grabiec [7] provided a generalization of Jungck's common fixed point theorem [8] for these spaces. In doing so, they introduced notions like weakly commuting and R −weakly commuting mappings and formulated intuitionistic fuzzy analogs of existing theorems, including Pant's theorem. In this paper, we present specific conditions under which four self−mappings on a chainable intuitionistic fuzzy metric space possess a unique common fixed point.

2 Preliminaries

Definition 2.1: A binary operation $*: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t −norm if $*$ is satisfying the following conditions

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.2: A binary operation $\lozenge: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t −conorm if $\lozenge$ is satisfying the following conditions

(a) $\lozenge$ is commutative and associative;
(b) $\lozenge$ is continuous;
(c) $a \lozenge 0 = a$ for all $a \in [0,1]$;
(d) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.3: A 5−tuple $(X, \mathcal{M}, N, *, \lozenge)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$ norm, $\lozenge$ is a continuous t −conorm and $\mathcal{M}, N$ are fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $\mathcal{M}(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
(ii) $\mathcal{M}(x, y, 0) = 0$ for all $x, y \in X$;
(iii) $\mathcal{M}(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(iv) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(v) $\mathcal{M}(x, y, t) * \mathcal{M}(y, z, s) \leq \mathcal{M}(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(vi) $\mathcal{M}(x, y, .): (0, \infty) \longrightarrow [0,1]$ is left continuous for all $x, y \in X$;
(vii) $\lim_{t\to\infty} \mathcal{M}(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;
(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(xi) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(xii) $N(x, y, .): (0, \infty) \longrightarrow [0,1]$ is right continuous for all $x, y \in X$;
(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$;

Then $(\mathcal{M}, N)$ is called an intuitionistic fuzzy metric on X. The functions $\mathcal{M}(x, y, t)$and $N(x, y, t)$denote the degree of nearness between x and y with respect to t, respectively.

Example 2.1: Let $X = \mathbb{N}$. Define $a * b = max\{0, a + b - 1\}$ and

$a \lozenge b = a + b - ab$ for all$a, b \in [0,1]$ and let $\mathcal{M}$ and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}(x, y, t) = \begin{cases} \frac{x}{y}, & if\ x \leq y \\ \frac{y}{x}, & if\ y \leq x \end{cases}$$

$$N(x, y, t) = \begin{cases} \frac{y - x}{y}, & if\ x \leq y \\ \frac{x - y}{x}, & if\ y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}, N, *, \lozenge)$ is an intuitionistic fuzzy metric space.

Definition 2.4: An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively of u in $\mathcal{A}_{\zeta,\eta}$ and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Remark 2.1: An intuitionistic fuzzy metric spaces with continuous t −norm $*$ and continuous t −conorm $\lozenge$ defined by $a * a \geq a$ and $(1 - a) \lozenge (1 - a) \leq (1 - a)$ for all $a \in [0,1]$. Then for all $x, y \in X, \mathcal{M}(x, y,)$ is non decreasing and $N(x, y, \lozenge)$ is non increasing.

Remark 2.2: Inintuitionistic fuzzy metric space$X, \mathcal{M}(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Definition 2.5: Let $(X, \mathcal{M}, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for all $t > 0$, and $p > 0$,
$$\lim_{n\to\infty} \mathcal{M}\left(x_{n+p}, x_n, t\right) = 1, \lim_{n\to\infty} N\left(x_{n+p}, x_n, t\right) = 0.$$

(ii) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$, $\lim_{n\to\infty} \mathcal{M}(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$.

Since $*$ and $\lozenge$ are continuous, the limit is uniquely determined from $(\boldsymbol{v})$ **and** $(\boldsymbol{xi})$ **of definition 2.3**, respectively.

Definition 2.6: An intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \lozenge)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.7: An intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \lozenge)$ is said to be compact if every sequence in X contains a convergent subsequence.

Lemma 2.1: Let $(X, \mathcal{M}, N, *, \lozenge)$ be an intuitionistic fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0, 1)$,

$\mathcal{M}(x,y,kt) \geq \mathcal{M}(x,y,t)$ and $N(x,y,kt) \leq N(x,y,t)$ then $x = y$.

Proof. Since $\mathcal{M}(x,y,kt) \geq \mathcal{M}(x,y,t)$, and $N(x,y,kt) \leq N(x,y,t)$,

we have $\mathcal{M}(x,y,t) \geq \mathcal{M}(x,y,\frac{t}{k})$ and $N(x,y,t) \leq N\left(x,y,\frac{t}{k}\right)$.

By repeated application of above inequalities, we have

$$\mathcal{M}(x,y,t) \geq \mathcal{M}(x,y,\frac{t}{k}) \geq \mathcal{M}(x,y,\frac{t}{k^2}) \geq \ldots \geq \mathcal{M}(x,y,\frac{t}{k^n}) \geq \cdots$$

And $N(x,y,t) \leq N(x,y,\frac{t}{k}) \leq N(x,y,\frac{t}{k^2}) \leq \ldots \leq N(x,y,\frac{t}{k^n}) \leq \cdots$

For $n \in \mathbb{N}$, which tends to 1 and 0 as $n \to \infty$, respectively.

Thus $\mathcal{M}(x,y,t) = 1$ and $N(x,y,t) = 0$ for all $t > 0$ and we get $x = y$.

Lemma 2.2: Let $(X,\mathcal{M},N,*,\Diamond)$ be an intuitionistic fuzzy metric space and

$\{y_n\}$ be a sequence in X. If there exists a number $k \in (0,1)$such that

$$\mathcal{M}(y_{n+2},y_{n+1},kt) \geq \mathcal{M}(y_{n+1},y_n,t),$$
$$N(y_{n+2},y_{n+1},kt) \leq N(y_{n+1},y_n,t) \qquad \ldots (1)$$

for all $t > 0$ and $n = 1,2,\ldots$, then $\{y_n\}$ is a Cauchy sequence in X.

Proof. By simple induction with the condition (1) with the help of

C. Alaca et al. [2], we have for all $t > 0$ and $n = 0,1,2,\cdots$,

$$\mathcal{M}(y_{n+1},y_{n+2},t) \geq \mathcal{M}\left(y_1,y_2,\frac{t}{k^n}\right),$$
$$N(y_{n+1},y_{n+2},t) \leq N\left(y_1,y_2,\frac{t}{k^n}\right) \qquad \ldots (2)$$

Thus by (2) and using definition **2.3**(($\boldsymbol{v}$) **and** ($\boldsymbol{xi}$)), for any positive integer p and real number $t > 0$, we have

$$\mathcal{M}(y_n,y_{n+p},t) \geq \mathcal{M}\left(y_n,y_{n+1},\frac{t}{p}\right)^{p-times} * \cdots * \mathcal{M}(y_{n+p-1}y_{n+p},\frac{t}{p})$$
$$\geq \mathcal{M}(y_1,y_2,\frac{t}{pk^{n-1}})^{p-times} * \cdots * \mathcal{M}(y_1,y_2,\frac{t}{pk^{n+p-2}})$$

and

$$N(y_n,y_{n+p},t) \leq N\left(y_n,y_{n+1},\frac{t}{p}\right)^{p-times} * \cdots * N(y_{n+p-1}y_{n+p},\frac{t}{p})$$
$$\leq N(y_1,y_2,\frac{t}{pk^{n-1}})^{p-times} * \cdots * N\left(y_1,y_2,\frac{t}{pk^{n+p-2}}\right).$$

Therefore, by definition **2.3**($(\boldsymbol{vii})$ **and** $(\boldsymbol{xiii})$), we have

$$\lim_{n\to\infty} \mathcal{M}\left(y_n, y_{n+p}, t\right) \geq 1^{p-times} * \cdots * 1 \geq 1.$$

and

$$\lim_{n\to\infty} \mathrm{N}\left(y_n, y_{n+p}, t\right) \leq 0^{p-times} \lozenge \cdots \lozenge\, 0 \leq 0.$$

Which implies that $\{y_n\}$ is a Cauchy sequence in X. This completes the proof.

Alaca,Turkoglu and Yildiz [3], introduced the notions of compatible mappings in intuitionistic fuzzy metric space, akin to the concept of compatible mappings introduced by Jungck, G. [8], in metric spaces as follows:

Definition 2.8: A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \lozenge)$ is said to be compatible

if $\lim_{n\to\infty} \mathcal{M}\,(fgx_n, gfx_n, t) = 1$, and $\lim_{n\to\infty} N\,(fgx_n, gfx_n, t) = 0$ for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.9: A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \lozenge)$ is said to be non compatible if $\lim_{n\to\infty} \mathcal{M}\,(fgx_n, gfx_n, t) \neq 1$, or non-existent and $\lim_{n\to\infty} N\,(fgx_n, gfx_n, t) \neq 0$ or non-existent for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.10: Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Definition 2.11: Let $(X, \mathcal{M}, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. A finite sequence $x = x_0, x_1, x_2, \cdots, x_n = y$ is called $\in -$chain from x to y if there exists a positive number $\in > 0$ such that $\mathcal{M}(x_i, x_{i-1}, t) > 1-\in$ and $N(x_i, x_{i-1}, t) > 1-\in$ for all $t > 0$ and $i = 1,2, \cdots, n$.

An intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \lozenge)$ is called $\in -$chainable

if for any $x, y \in X$, there exists an $\in -$chain from x to y.

3 The Main Results

Theorem 3.1: Let A, B, S, and T be self maps of a complete $\in -$chainable intuitionistic fuzzy metric spaces $(X, \mathcal{M}, N, *, \lozenge)$ with continuous $t -$norm $*$ and continuous $t -$conorm $\lozenge$ defined by

$a * a \geq a$ and $(1-a) \lozenge (1-a) \leq (1-a)$ for all $a \in [0,1]$ Satisfying the following condition:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,
(2) A and S are continuous,
(3) The pairs (A, S) and (B, T) are weakly compatible,

(4) There exist $q \in (0,1)$ such that

$$\mathcal{M}(Ax,By,qt) \geq \left\{\mathcal{M}(Sx,Ty,t) * \mathcal{M}(Sx,Ax,t) * \frac{a\mathcal{M}(Ax,Ty,t)+b\mathcal{M}(Ty,Sx,t)}{a+b} * \mathcal{M}(By,Ty,t) * \mathcal{M}(Ax,Ty,t) * \mathcal{M}(Sx,By,t)\right\}$$

and

$$N(Ax,By,qt) \leq \left\{N(Sx,Ty,t) \lozenge N(Sx,Ax,t) \lozenge \frac{aN(Ax,Ty,t)+bN(Ty,Sx,t)}{a+b} \lozenge N(By,Ty,t) \lozenge N(Ax,Ty,t) \lozenge N(Sx,By,t)\right\}$$

For every $x,y \in X$ and $t > 0 >$, where $a,b \geq 0$ with $a\&b$ cannot be simultaneously 0. Then A,B,S and T have a unique common fixed point in X.

Proof. As $A(X) \subseteq T(X)$, for any $x_0 \in X$, there exist a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for this point $x_1,$ we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we can find a sequence $\{y_n\}$ in X as follows:

$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1,2,\cdots$

By Theorem of Alaca et al. [2] we can conclude that $\{y_n\}$ in X.

Since X is complete, therefore sequence $\{y_n\}$ in X converges to z for some z in X and so the sequences $\{Tx_{2n-1}\}, \{Ax_{2n-2}\}, \{Sx_{2n}\}$ and $\{Bx_{2n-1}\}$ also converges to z.

Since X is $\in -$chainable, there exists $\in -$chain from x_n to x_{n+1}, that is there exists a finite sequence

$x_n = y_1, y_2, \cdots, y_l = x_{n+1}$

such that $\mathcal{M}(y_i, y_{i-1}, t) > 1{-}\in$

and $N(y_i, y_{i-1}, t) < 1{-}\in$ for all $t > 0$ and $i = 1,2,\cdots,l$.

So we have

$$\mathcal{M}(x_n,x_{n+1},t) \geq \mathcal{M}\left(y_1,y_2,\frac{t}{l}\right) * \mathcal{M}\left(y_2,y_3,\frac{t}{l}\right) * \cdots * \mathcal{M}\left(y_{l-1},y_l,\frac{t}{l}\right) > (1{-}\in) * (1{-}\in) * \cdots * (1{-}\in) \geq (1{-}\in)$$

and

$$N(x_n,x_{n+1},t) \leq N\left(y_1,y_2,\frac{t}{l}\right) \lozenge N\left(y_2,y_3,\frac{t}{l}\right) \lozenge \cdots \lozenge N\left(y_{l-1},y_l,\frac{t}{l}\right) < (1{-}\in) \lozenge (1{-}\in) \lozenge \cdots \lozenge (1{-}\in) \leq (1{-}\in)$$

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For $m > n$,

$$\mathcal{M}(x_n, x_m, t) \geq \mathcal{M}(x_n, x_{n+1}, \frac{t}{m-n}) * \mathcal{M}(x_{n+1}, x_{n+2}, \frac{t}{m-n})$$

$$* \cdots * \mathcal{M}(x_{m-1}, x_m, \frac{t}{m-n}) > (1-\epsilon) * (1-\epsilon) * \cdots * (1-\epsilon) \geq (1-\epsilon) \text{ and}$$

$$N(x_n, x_m, t) \leq N(x_n, x_{n+1}, \frac{t}{m-n}) \lozenge N(x_{n+1}, x_{n+2}, \frac{t}{m-n})$$

$$\lozenge \cdots \lozenge \mathcal{M}(x_{m-1}, x_m, \frac{t}{m-n}) < (1-\epsilon) \lozenge (1-\epsilon) \lozenge \cdots \lozenge (1-\epsilon) \leq (1-\epsilon)$$

Therefore $\{x_n\}$ is a Cauchy sequence in X and hence there exists x in X such that $x_n \to x$. from condition (2) $Ax_{2n-2} \to Ax$, $Sx_{2n} \to Sx$ as limit $n \to \infty$. By uniqueness of limits, we have $Ax = z = Sx$. Since pair (A, S) is weakly compatible, therefore, $ASx = SAx$ and so $Az = Sz$.

from condition (2) we have $ASx_{2n} \to ASx$ and therefore, $ASx_{2n} \to Sz$.

Also from continuity of S, we have $SSx_{2n} \to Sz$.

from condition (4), we get

$$\mathcal{M}(ASx_{2n}, Bx_{2n-1}, qt) \geq \Big\{\mathcal{M}(SSx_{2n}, Tx_{2n-1}, t) * \mathcal{M}(SSx_{2n}, ASx_{2n}, t) * \frac{a\mathcal{M}(ASx_{2n}, Tx_{2n-1}, t) + b\mathcal{M}(Tx_{2n-1}, SSx_{2n}, t)}{a+b} * \mathcal{M}(Bx_{2n-1}, Tx_{2n-1}, t) * \mathcal{M}(ASx_{2n}, Tx_{2n-1}, t) * \mathcal{M}(SSx_{2n}, Bx_{2n-1}, t)\Big\}$$

and

$$N(ASx_{2n}, Bx_{2n-1}, qt) \leq \Big\{N(SSx_{2n}, Tx_{2n-1}, t) \lozenge N(SSx_{2n}, ASx_{2n}, t) \lozenge \frac{aN(ASx_{2n}, Tx_{2n-1}, t) + bN(Tx_{2n-1}, SSx_{2n}, t)}{a+b} \lozenge N(Bx_{2n-1}, Tx_{2n-1}, t) \lozenge N(ASx_{2n}, Tx_{2n-1}, t) \lozenge N(SSx_{2n}, Bx_{2n-1}, t)\Big\}$$

Proceeding limit as $n \to \infty$, we have

$$\mathcal{M}(Sz, z, qt) \geq \Big\{\mathcal{M}(Sz, z, t) * \mathcal{M}(Sz, Sz, t) * \frac{a\mathcal{M}(Sz, z, t) + b\mathcal{M}(z, Sz, t)}{a+b} * \mathcal{M}(z, z, t) * \mathcal{M}(Sz, z, t) * \mathcal{M}(Sz, z, t)\Big\}$$

and

$$N(Sz, z, qt) \leq \left\{ N(Sz, z, t) \lozenge N(Sz, Sz, t) \lozenge \frac{aN(Sz, z, t) + bN(z, Sz, t)}{a + b} \lozenge N(z, z, t) \lozenge N(Sz, z, t) \lozenge N(Sz, z, t) \right\}.$$

From **lemma 2.1**, we get $Sz = z$, and hence $Az = Sz = z$.

Since $A(X) \subseteq T(X)$, there exists v in X such that $Tv = Az = z$.

from condition (4), we have

$$\mathcal{M}(Ax_{2n}, Bv, qt) \geq \left\{ \mathcal{M}(Sx_{2n}, Tv, t) * \mathcal{M}(Sx_{2n}, Ax_{2n}, t) * \frac{a\mathcal{M}(Ax_{2n}, Tv, t) + b\mathcal{M}(Tv, Sx_{2n}, t)}{a + b} * \mathcal{M}(Bv, Tv, t) * \mathcal{M}(Ax_{2n}, Tv, t) * \mathcal{M}(Sx_{2n}, Bv, t) \right\}$$

and

$$N(Ax_{2n}, Bv, qt) \leq \left\{ N(Sx_{2n}, Tv, t) \lozenge N(Sx_{2n}, Ax_{2n}, t) \lozenge \frac{aN(Ax_{2n}, Tv, t) + bN(Tv, Sx_{2n}, t)}{a + b} \lozenge N(Bv, Tv, t) \lozenge N(Ax_{2n}, Tv, t) \lozenge N(Sx_{2n}, Bv, t) \right\}$$

Letting $n \to \infty$, we have

$$\mathcal{M}(z, Bv, qt) \geq \left\{ \mathcal{M}(z, Tv, t) * \mathcal{M}(z, z, t) * \frac{a\mathcal{M}(z, Tv, t) + b\mathcal{M}(Tv, z, t)}{a + b} * \mathcal{M}(Bv, Tv, t) * \mathcal{M}(z, Tv, t) * \mathcal{M}(z, Bv, t) \right\}$$

$$= \left\{ \mathcal{M}(z, z, t) * \mathcal{M}(z, z, t) * \frac{a\mathcal{M}(z, z, t) + b\mathcal{M}(z, z, t)}{a + b} * \mathcal{M}(Bv, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(z, Bv, t) \right\} \geq \mathcal{M}(Bv, z, t)$$

and

$$N(z,Bv,qt) \leq \left\{N(z,Tv,t) \lozenge N(z,z,t) \lozenge \frac{aN(z,Tv,t) + bN(Tv,z,t)}{a+b} \lozenge N(Bv,Tv,t) \lozenge N(z,Tv,t) \lozenge N(z,Bv,t)\right\}$$

$$= \left\{N(z,z,t) \lozenge N(z,z,t) \lozenge \frac{aN(z,z,t) + bN(z,z,t)}{a+b} \lozenge N(Bv,z,t) \lozenge N(z,z,t) \lozenge N(z,Bv,t)\right\} \leq N(Bv,z,t).$$

By **lemma 2.1**, we have $Bv = z$, and therefore, we have $Tv = Bv = z$.

Since (B,T) is weakly compatible, therefore, $TBv = BTv$ and

hence $Tz = Bz$.

from condition (4), we have

$$\mathcal{M}(Ax_{2n},Bz,qt) \geq \left\{\mathcal{M}(Sx_{2n},Tz,t) * \mathcal{M}(Sx_{2n},Ax_{2n},t) * \frac{a\mathcal{M}(Ax_{2n},Tz,t) + b\mathcal{M}(Tz,Sx_{2n},t)}{a+b} * \mathcal{M}(Bz,Tz,t) * \mathcal{M}(Ax_{2n},Tz,t) * \mathcal{M}(Sx_{2n},Bz,t)\right\}$$

and

$$N(Ax_{2n},Bz,qt) \leq \left\{N(Sx_{2n},Tz,t) \lozenge N(Sx_{2n},Ax_{2n},t) \lozenge \frac{aN(Ax_{2n},Tz,t) + bN(Tz,Sx_{2n},t)}{a+b} \lozenge N(Bz,Tz,t) \lozenge N(Ax_{2n},Tz,t) \lozenge N(Sx_{2n},Bz,t)\right\}.$$

Letting $n \to \infty$, we have

$$\mathcal{M}(z,Bz,qt) \geq \left\{\mathcal{M}(z,Tz,t) * \mathcal{M}(z,z,t) * \frac{a\mathcal{M}(z,Tz,t) + b\mathcal{M}(Tz,z,t)}{a+b} * \mathcal{M}(Bz,Tz,t) * \mathcal{M}(z,Tz,t) * \mathcal{M}(z,Bz,t)\right\}$$

$$= \left\{\mathcal{M}(z,z,t) * \mathcal{M}(z,z,t) * \frac{a\mathcal{M}(z,z,t) + b\mathcal{M}(z,z,t)}{a+b} * \mathcal{M}(Bz,z,t) * \mathcal{M}(z,z,t) * \mathcal{M}(z,Bz,t)\right\} \geq \mathcal{M}(Bz,z,t)$$

and

$$N(z,Bz,qt) \leq \left\{N(z,Tz,t) \lozenge N(z,z,t) \lozenge \frac{aN(z,Tz,t)+bN(Tz,z,t)}{a+b} \lozenge N(Bz,Tz,t) \lozenge N(z,Tz,t) \lozenge N(z,Bz,t)\right\}$$

$$= \left\{N(z,z,t) \lozenge N(z,z,t) \lozenge \frac{aN(z,z,t)+bN(z,z,t)}{a+b} \lozenge N(Bz,z,t) \lozenge N(z,z,t) \lozenge N(z,Bz,t)\right\} \leq N(Bz,z,t).$$

Which implies that $Bz = z$.

Therefore, $Az = Sz = Bz = Tz = z$.

Hence A, B, S and T have a unique common fixed point in X.

For uniqueness, let w be another common fixed point of A, B, S and T.

Then from condition (4), we have

$$\mathcal{M}(z,w,qt) = \mathcal{M}(Az,Bw,qt) \geq \left\{\mathcal{M}(Sz,Tw,t) * \mathcal{M}(Sz,Az,t) * \frac{a\mathcal{M}(Az,Tw,t)+b\mathcal{M}(Tw,Sz,t)}{a+b} * \mathcal{M}(Bw,Tw,t) * \mathcal{M}(Az,Tw,t) * \mathcal{M}(Sz,Bw,t)\right\} \geq \mathcal{M}(z,w,t)$$

and

$$N(z,w,qt) = N(Az,Bw,qt) \leq \left\{N(Sz,Tw,t) \lozenge N(Sz,Az,t) \lozenge \frac{aN(Az,Tw,t)+bN(Tw,Sz,t)}{a+b} \lozenge N(Bw,Tw,t) \lozenge N(Az,Tw,t) \lozenge N(Sz,Bw,t)\right\} \leq N(z,w,t).$$

From **lemma 2.1**, we conclude that $z = w$.

Hence A, B, S and T have a unique common fixed point in X.

4 Conclusion

In this paper, we establish certain conditions under which four self−mappings on an $\in$ −chainable intuitionistic fuzzy metric space admit a unique common fixed point. This study can be further expanded by considering a greater number of self−mappings and formulating fixed point theorems within broader and more generalized frameworks.

5 References

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